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THE FLOW OF GAS IN CHANNELS IN THE PRESENCE OF HEAT TRANSFER

by M. M. Nazarchuk

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THE FLOW OF GAS IN CHANNELS IN THE PRESENCE OF HEAT TRANSFER

By M. M. Nazarchuk

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The book presents the one-dimensional theory of the flow of gas in channels in the presence of heat transfer between walls and gas.

New methods of calculation based on the theory of the boundary layer are given for the flow of gas in tubes; the conditions of crisis in such flows are examined, and the limits of applicability of Reynolds analogies are indicated.

The book is directed to scientific workers and engineers concerned with applied gas dynamics. It is also of value as a textbook for graduate workers and students.

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FLOW OF GAS IN CHANNELS IN THE PRESENCE OF HEAT TRANSFER

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FOREWORD

The theory of the flow of gas in tubes in the presence of heat transfer is of great importance in several branches of heat engineering. The problem must be studied in order to solve many important technical problems concerning gas turbines, jet engines, nuclear reactors, and so on. Here I consider some new methods of solving for the flow of gas in tubes in the presence of heat transfer; one- and two-dimensional models are used.

In sections 1-4 of chapter I deal briefly with the results from the one-dimensional theory that are required in later chapters. The one-dimensional theory is to be found expounded in more detail in specialized works such as [1, 3, 5, 39].

Chapters II and III deal in more detail with planar flow between two parallel walls having symmetrical heating.

Chapter II also gives the method of calculation for a circular tube.

Chapter IV presents some results of the calculations.

In every case where no special comment is made, it is assumed that heat enters the gas; the methods given here cannot, in general, be used if the gas loses heat.

Relationships for the axis of the tube are used here together with integral relationships, in which we have a difference from existing integral methods. This makes possible a more detailed study of the behavior of the velocity and temperature profiles; here the ordinary power-law profiles are characterized not only by the powers but also by the curvature at the axis. Calculations carried out in this way have enabled me to derive some aspects of the behavior of the velocity and temperature profiles that are essentially undetectable by the integral methods.

The MKS system of units (GOST 9867-61) is used; the basic quantities are the meter, the kilogram (mass), the second, and the ^{o}K.

It is assumed that the reader is acquainted with engineering thermodynamics and with the principles of boundary layer theory.

All the calculations are for air, on the assumption that k(the isentropy constant) and the heat capacity c_p are independent of temperature.

The calculations in chapter IV were performed with an electronic digital computer at the Institute of Cybernetics, Academy of Sciences, Ukrainian SSR. For this I am indebted to the following of my colleagues there: V. Ye. Shamanskiy, Ye. L. Yushchenko, V. I. Khil'chenko, and I. S. Markova.

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NOMENCLATURE

Subscripts

i	initial section
f	final section
0	wall of channel
1	axis of channel
m	mean-mass quantities

Quantities

a	speed of sound, m/sec
a_*	critical speed, m/sec
c_p	thermal capacity at constant pressure, J/kg-°K
c_v	thermal capacity at constant volume, J/kg-°K
D	hydraulic diameter of channel, m
F	cross-sectional area of channel (m^2) in chapter I; a parameter in chapters II-IV
g	gas-dynamic function (reduced flow)
G	mass flow of gas, kg/sec
h	half the width of a plane-parallel channel, m
i	enthalpy, J/kg
k	isentropy parameter
L	length of channel, m
m	form parameter of the retardation-temperature profile
M	Mach number
M_*	velocity coefficient ($M_* = w/a_*$)
n	form parameter of velocity profile
N	power, W
p	pressure, N/m^2
P	retardation pressure, N/m^2
Pr	Prandtl number
Q	amount of heat, J/kg

q	specific thermal flux, J/m^2-sec
R	gas constant, J/kg-$^{\circ}$K
Re	Reynolds number
S'_h	heated perimeter, m
S_w	wetted perimeter, m
St	Stanton number
T	temperature, $^{\circ}$K
u	lengthwise velocity component, m/sec
v	transverse velocity component, m/sec
V	ratio of flow velocity to limiting velocity
w	velocity of one-dimensional flow, m/sec
w_l	limiting velocity m/sec
x	lengthwise coordinate, m
y	transverse coordinate, m
γ	gas-dynamic function
α	heat-transfer factor, J/m^2-sec-$^{\circ}$K
β	power in the relation of viscosity to temperature
δ	thickness of viscous sublayer, m
η	Dorodnitsyn's variable, m
H	ratio of η to the value of this at the axis ($H = \eta/\eta_1$)
ϵ	coefficient of turbulent transfer, N-sec/m^2
λ	thermal conductivity, J/m-sec-$^{\circ}$K
μ	dynamic viscosity, N-sec/m^2
ν	kinetic viscosity, m^2/sec
Π	gas-dynamic function
Θ	retardation temperature, $^{\circ}$K
φ	local angle of widening of channel
χ	gas-dynamic function
ζ	resistance coefficient
ρ	density, kg/m^3
τ	gas-dynamic function

CHAPTER I

ONE-DIMENSIONAL THEORY

Here I consider steady-state flows in channels subject to loss and to heat transfer between walls and gas. It is in every case assumed that the gas does no external work. The speed of the gas at the inlet (in the initial section) is taken to be subsonic if no special mention is made. The gas is assumed perfect, so at any point the pressure p, density ρ, and temperature T are related by the equation of state:

$$p = \rho RT. \tag{1.1}$$

The mass of gas flowing through the cross-section (area F) in a second is $G = \rho wF$, in which w is the velocity. G is constant at all points along the channel in the steady state*:

$$G = \rho wF = \text{const.} \tag{1.2}$$

This is called the equation of continuity.

To (1.1) and (1.2) we must add relations for the energy conversion and for the momentum.

The energy equation is given by thermodynamics as

$$dQ = di + d\left(\frac{w^2}{2}\right), \tag{1.3}$$

in which dQ is the heat entering the gas from outside, di and $d(w^2/2)$ being respectively the changes in enthalpy and kinetic energy for an

*Here and subsequently it is assumed that there are no sources or sinks for the gas.

element of the gas*.

The Ox axis lies along the midline of the channel downstream; $i = c_p T$ for a perfect gas, and we assume that the thermal capacity c_p is not dependent on T. Then instead of (1.3) we have

$$\frac{dQ}{dx} = c_p \frac{d}{dx}\left(T + \frac{w^2}{2c_p}\right). \qquad (1.4)$$

We introduce the symbols

$$q = \frac{G}{S_h} \frac{dQ}{dx}, \qquad (1.5$$

$$\Theta = T + \frac{w^2}{2c_p}, \qquad (1.6)$$

in which S_h is the heated perimeter.

We substitute (1.5) and (1.6) into (1.4) to get

$$\frac{d\Theta}{dx} = \frac{S_h}{Gc_p} q. \qquad (1.7)$$

Here q is the amount of heat entering the gas every second through one square meter of the heated surface; it is called the specific heat

*Here i and Q relate to 1 kg of gas.

flux.

Also, Θ is called the retardation temperature. No heat transfer ($q = 0$) implies Θ = constant from (1.7), i.e., from (1.6)

$$\Theta = T + \frac{w^2}{2c_p} = \text{const.}$$

This temperature is thus reached when the velocity falls to zero in a flow that is isolated as far as energy is concerned.

An influx of heat ($q > 0$) causes this temperature to increase along the flow ($d\Theta/dx > 0$), while heat loss ($q < 0$) implies a fall ($d\Theta/dx < 0$).

The system represented by (1.1), (1.2), (1.6), and (1.7) is closed by adding the equation for the momentum (the general Bernoulli equation):

$$\frac{1}{\rho}\frac{dp}{dx} + w\frac{dw}{dx} + \zeta\frac{w^2}{2D} = 0, \tag{1.8}$$

which relates the pressure p to the speed w; here D is the hydraulic diameter of the channel:

$$D = \frac{4F}{S_w}. \tag{1.9}$$

The quantity ζ is an integral characteristic dependent mainly on Re, apart from several other factors. There is little experimental evidence on the precise relation of ζ to heat-transfer conditions, surface roughness, and so on, so it is often assumed (especially for rough calculations) that ζ is constant at all points, some mean value being taken.

The system consisting of (1.1), (1.2), (1.6), (1.7), and (1.8) contains the five unknowns T, p, ρ w, and Θ. It is assumed that G and c_p are known constants.

Also, it is assumed that q, the cross-sectional area F, and the heated and wetted perimeters S_h and S_w are known functions of x*, so (1.7) gives directly Θ for any point in the channel as

$$\Theta = \frac{1}{Gc_p} \int_0^x qS_h\, dx + \Theta_c . \tag{1.10}$$

The remaining four equations are conveniently reduced to one in the unknown

$$M^2 = \frac{w^2}{a^2} .$$

This M is the Mach number. For a perfect gas

$$a^2 = kRT, \tag{1.11}$$

in which

---

*The quantity actually given is often T_0 (wall temperature) rather than q. Then we need to know a further experimental coefficient in the one-dimensional theory apart from ζ, namely the heat-transfer factor α, which is related to q by $q = \alpha(T - \Theta)$.

$$k = \frac{c_p}{c_v},$$

so

$$M^2 = \frac{w^2}{kRT}. \tag{1.12}$$

We use

$$c_p - c_v = R,$$

to find that

$$c_p = \frac{k}{k-1} R. \tag{1.13}$$

In place of (1.6) we have

$$\Theta = T + \frac{k-1}{2k} \frac{w^2}{R}, \tag{1.14}$$

and so from (1.12)

$$\frac{\Theta}{T} = 1 + \frac{k-1}{2} M^2. \tag{1.15}$$

We use (1.1) and (1.12) to put (1.8) in the form

$$\frac{1}{p}\frac{dp}{dx} + \frac{kM^2}{w}\frac{dw}{dx} + \zeta \frac{kM^2}{2D} = 0. \tag{1.16}$$

Taking logarithms of (1.1) and differentiating, we have

$$\frac{1}{p}\frac{dp}{dx} = \frac{1}{T}\frac{dT}{dx} + \frac{1}{\rho}\frac{d\rho}{dx} . \tag{1.17}$$

Similarly from (1.2)

$$\frac{1}{\rho}\frac{d\rho}{dx} = - \frac{1}{w}\frac{dw}{dx} - \frac{1}{F}\frac{dF}{dx} . \tag{1.18}$$

(1.12) gives

$$\frac{1}{T}\frac{dT}{dx} = \frac{2}{w}\frac{dw}{dx} - \frac{1}{M^2}\frac{dM^2}{dx} . \tag{1.19}$$

Substituting (1.18) and (1.19) into (1.17), we have

$$\frac{1}{p}\frac{dp}{dx} = \frac{1}{w}\frac{dw}{dx} - \frac{1}{M^2}\frac{dM^2}{dx} - \frac{1}{F}\frac{dF}{dx} . \tag{1.20}$$

Substituting (1.20) into (1.16) we have

$$\frac{1 + kM^2}{w}\frac{dw}{dx} - \frac{1}{M^2}\frac{dM^2}{dx} + \frac{\zeta kM^2}{2D} - \frac{1}{F}\frac{dF}{dx} = 0. \tag{1.21}$$

From (1.12) and (1.15) we have

$$w^2 = kR\Theta \frac{M^2}{1 + \frac{k-1}{2}M^2},$$

so

$$\frac{1}{w}\frac{dw}{dx} = \frac{1}{2\Theta}\frac{d\Theta}{dx} + \frac{1}{2M^2\left(1 + \frac{k-1}{2}M^2\right)}\frac{dM^2}{dx}. \qquad (1.22)$$

This value for $(dw/dx)/w$ is substituted into (1.21) to give

$$\frac{1 - M^2}{2M^2\left(1 + \frac{k-1}{2}M^2\right)}\frac{dM^2}{dx} = \frac{1 + kM^2}{2\Theta}\frac{d\Theta}{dx} + \zeta\frac{kM^2}{2D} - \frac{1}{F}\frac{dF}{dx}. \qquad (1.23)$$

T is found from (1.15) after this equation has been integrated; w is found from (1.12), and p and ρ from (1.1) and (1.2) as

$$p = \frac{GRT}{wF}, \quad \rho = \frac{p}{RT}. \qquad (1.24)$$

A scale of measurement for the speed sometimes more convenient than the speed of sound is the critical speed, which is the flow velocity equal to the local speed of sound, which is usually denoted by a_*.

From (1.1) and (1.14) we have

$$a^2 + \frac{k-1}{2}w^2 = kR\Theta.$$

Putting here $w = a = a_*$, we have

$$a_*^2 = \frac{2k}{k+1} R\Theta. \tag{1.25}$$

The critical speed is constant in a flow isolated as regards energy. In general, this speed varies from one cross-section to another in accordance with the variation in Θ, as (1.25) shows.

We introduce the symbol

$$M_*^2 = \frac{w^2}{a_*^2} = \frac{k+1}{2kR}\frac{w^2}{\Theta}. \tag{1.26}$$

From (1.14) and (1.26) we have

$$\frac{T}{\Theta} = 1 - \frac{k-1}{k+1} M_*^2. \tag{1.27}$$

From (1.15) and (1.27) we have the relation of M to M_*:

$$M^2 = \frac{2}{k+1} \frac{M_*^2}{1 - \frac{k-1}{k+1} M_*^2}. \tag{1.28}$$

We substitute the M^2 of (1.28) into (1.23) to get

$$\frac{1 - M_*^2}{2M_*^2}\frac{dM_*^2}{dx} = \frac{1 + M_*^2}{2\Theta}\frac{d\Theta}{dx} + \frac{\zeta k}{k+1}\frac{M_*^2}{D} - \frac{1 - \frac{k-1}{k+1} M_*^2}{F}\frac{dF}{dx}. \tag{1.29}$$

T is determined from (1.27) after (1.29) has been integrated; w is found from (1.26), and the pressure and density from (1.24).

It is often convenient to use the retardation pressure as well as the retardation temperature; this is given by the adiabatic equation

$$\frac{P}{p} = \left(\frac{\Theta}{T}\right)^{\frac{k}{k-1}},$$

so from (1.15)

$$\frac{P}{p} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}}, \tag{1.30}$$

and from (1.25)

$$\frac{p}{P} = \left(1 - \frac{k-1}{k+1} M_*^2\right)^{\frac{k}{k-1}}. \tag{1.31}$$

Pressure p takes the value P when the speed of an isentropic flow falls to zero.

The functions

$$\tau = 1 \frac{k-1}{k+1} M_*^2 = \frac{1}{1 + \frac{k-1}{2} M^2}, \tag{1.32}$$

$$\Pi = \tau^{\frac{k}{k-1}}, \tag{1.33}$$

are respectively the ratio of the temperature to Θ and the ratio of the pressure to P; these and some other functions (which are called gas-dynamic ones) have been tabulated, which often facilitates calculations*.

We introduce two further gas-dynamic functions that will be needed subsequently; they are convenient in calculations relating to the flow rate G.

From (1.24)

$$G = \frac{pwF}{RT},$$

so, from (1.26), (1.27), and (1.32)

$$\frac{G}{F} = \sqrt{\frac{2k}{(k+1)R}}\,\frac{p}{\sqrt{\Theta}}\,\frac{M_*}{\tau}\,. \tag{1.34}$$

Expressing p in terms of P, we have from (1.31)-(1.33) that

$$\frac{G}{F} = \sqrt{\frac{2k}{(k+1)R}}\,\frac{P}{\sqrt{\Theta}}\,M_*\,\frac{\Pi}{\tau}\,. \tag{1.35}$$

(1.35) shows that G/F (flow per unit area) has a maximum for $M_* =$ $= 1$ for any fixed P and Θ; this is termed the critical flow and is denoted by $(G/F)_*$. From (1.35)

*Tables of these functions are to be found in [14, 16], for example. Appendix III gives tables of these functions for air (k = 1.4).

$$\left(\frac{G}{F}\right)_* = \sqrt{\frac{2k}{(k+1)\ R}}\ \frac{P}{\sqrt{\Theta}} \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}}.$$

The ratio

$$g = \frac{G}{F} : \left(\frac{G}{F}\right)_* = \left(\frac{k+1}{2}\right)^{\frac{1}{k-1}} M_* \left(1 - \frac{k-1}{k+1} M_*^2\right)^{\frac{1}{k-1}} \qquad (1.36)$$

is called the referred (reduced) flow rate*.

Then from (1.35) we have

$$G = j\ \frac{PF}{\sqrt{\Theta}}\ g, \qquad (1.37)$$

in which

$$j = \sqrt{\frac{k}{R} \left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}}.$$

For air ($k = 1.4$, $R = 288$ J/kg-°K) we have $j = 0.0404$.

The flow rate is expressed in terms of P in (1.37); it is often desirable to have it expressed in terms of p.

We introduce the function

---

*This is usually denoted by q in gas dynamics, but here q represents the specific heat flux.

$$Y = \frac{g}{\Pi} = \left(\frac{k+1}{2}\right)^{\frac{1}{k-1}} \frac{M_*}{1 - \frac{k-1}{k+1} M_*^2}, \tag{1.38}$$

and get from (1.34) that

$$G = j \frac{pF}{\sqrt{\Theta}} Y. \tag{1.39}$$

Equations (1.23) and (1.29) enable us to establish the effects of friction, channel configuration, and heat inflow on M and M_*.

Analogous equations for the velocity, temperature, pressure, and density are also of value. From (1.22) and (1.23) we have

$$\frac{1 - M^2}{w} \frac{dw}{dx} = \frac{2 + (k-1)M^2}{2\Theta} \frac{d\Theta}{dx} + \zeta \frac{kM^2}{2D} - \frac{1}{F} \frac{dF}{dx}. \tag{1.40}$$

From (1.16) and (1.40)

$$\frac{1 - M^2}{p} \frac{dp}{dx} = kM^2 \left(- \frac{2 + (k-1)M^2}{2\Theta} \frac{d\Theta}{dx} - \zeta \frac{1 + (k-1)M^2}{2D} + \frac{1}{F} \frac{dF}{dx} \right). \tag{1.41}$$

From (1.18) and (1.40)

$$\frac{1 - M^2}{\rho} \frac{d\rho}{dx} = - \frac{2 + (k-1)M^2}{2\Theta} \frac{d\Theta}{dx} - \zeta \frac{kM^2}{2D} + \frac{M^2}{F} \frac{dF}{dx}. \tag{1.42}$$

From (1.10) and (1.40)

$$\frac{1 - M^2}{T}\frac{dT}{dx} = (1 - kM^2)\frac{2 + (k - 1)M^2}{2\Theta}\frac{d\Theta}{dx} +$$

$$+ (k - 1)M^2\left(-\zeta\frac{kM^2}{2D} + \frac{1}{F}\frac{dF}{dx}\right). \qquad (1.43)$$

To conclude, we find the relation between the change in p, the rate of influx of heat, and ζ. From (1.30)

$$\frac{1}{p}\frac{dp}{dx} = \frac{1}{P}\frac{dP}{dx} - \frac{k}{2}\frac{1}{1 + \frac{k - 1}{2}M^2}\frac{dM^2}{dx}.$$

This value of $(dp/dx)/p$ is substituted into (1.16); (1.22) is used to give

$$\frac{1}{P}\frac{dP}{dx} = -\frac{kM^2}{2}\left(\frac{1}{\Theta}\frac{d\Theta}{dx} + \frac{\zeta}{D}\right). \qquad (1.44)$$

This shows that P decreases as x increases in adiatic processes $(d\Theta/dx = 0)$ and in ones involving influx of heat $(d\Theta/dx > 0)$.

The rate of decrease of P for a given M is dependent on the heat influx $(d\Theta/dx)$ and the rate of production of heat by friction (ζ).

This means that the variation in P may serve as a measure of the loss in processes of heat influx.

2. Qualitative Analysis: Principle of Reversal of Action and Flow Crisis

The following relationships are derived by a qualitative discussion of the equations. Firstly, $T/\Theta \geq 0$, so (1.27) implies that

$$M_*^2 \leq \frac{k+1}{k-1},$$

so $w^2 \leq w_1^2 = a_*^2(k+1)/(k-1)$, in which w_1 is the limiting velocity. From (1.25) we have

$$w_1^2 = \frac{2k}{k-1} R\Theta. \tag{1.45}$$

The speed at any point cannot be higher than the w_1 corresponding to that point.

The limiting speed varies from section to section when there is heat transfer, on account of the change in Θ, as in the case of the critical speed; w_1 is constant in a flow isolated from energy transfer and serves (together with the critical speed) as a convenient scale for the speeds.

We put

$$V = \frac{w}{w_1}, \tag{1.46}$$

From (1.45), (1.46), and (1.26) we have a relation of V to M_*:

$$V^2 = \frac{k-1}{k+1} M_*^2. \tag{1.47}$$

(1.27) and (1.47) imply that $V \to 1$ for $T \to 0$, so it is impossible to attain the limiting speed.

$V^2 = 1$ corresponds to $M_*^2 = (k + 1)/(k - 1) > 1$, so it is possible to approach the limiting speed only when the flow speed exceeds the local speed of sound. The transition through the speed of sound can be examined from (1.23). This shows that friction ($\zeta > 0$), heat influx ($d\Theta/dx > 0$), and reduction in cross-section ($dF/dx < 0$) cause M to rise if $M < 1$, whereas these factors have the opposite effect if $M > 1$. Similar deductions can be drawn about the speed, density, pressure, and temperature from (1.40)-(1.43). This is known as the principle of reversal of action [4].

Now we consider in more detail the effects associated with the two types of variation. We assume that $M < 1$ and that the over-all action is positive near the inlet:

$$\frac{1 + kM^2}{2\Theta}\frac{d\Theta}{dx} + \zeta\frac{kM^2}{2D} - \frac{1}{F}\frac{dF}{dx} > 0. \tag{1.48}$$

(1.23) implies that M must increase with x near the inlet.

There are several possible cases, which differ as regards the later behavior of the over-all action.

A. The over-all action becomes zero at $x = x'$, after which it becomes and remains negative.

If $M' = M|_{x=x'} < 1$, we have M maximal (M') at $x = x'$, there being a decrease for larger x (curve 1 of Fig. 1).

B. $M = 1$ at $x = x''$, and the over-all action becomes and remains negative. Then $M > 1$ for $x > x''$, so the speed increases from subsonic to supersonic (curve 2 of Fig. 1).

$$\frac{k + 1}{\Theta}\frac{d\Theta}{dx} + \frac{k\zeta}{D} - \frac{2}{F}\frac{dF}{dx} = 0, \tag{1.49}$$

C. If the over-all action is everywhere positive, we have $M \to 1$ possible only for $dM/dx \to +\infty$ (curve 3 of Fig. 1).

If $M \to 1$ at a finite distance $x = x'''$ from the input, the flow cannot extend continuously in the channel; if M were subsequently to decrease, we would have $dM^2/dx < 0$, so we would have $M^2 < 1$, and so

$1 - M^2 > 0$. If M were subsequently to increase, we would have $dM^2/dx > 0$, $M^2 > 1$, and $1 - M^2 < 0$. In both cases $(1 - M^2)(dM^2/dx)$ is negative, although the over-all action is positive, which from (1.23) conflicts with the assumed condition.

This impossibility of continuous extension is termed flow crisis.

(1.40)-(1.43) show that $dw/dx \to + \infty$ as well as $dM/dx \to + \infty$ when the crisis occurs, and also $dp/dx \to - \infty$, $d\rho/dx \to - \infty$, and $dT/dx \to - \infty$.

This means that there is a very rapid rise in M and in the speed near the crisis point, together with very rapid falls in pressure, density, and temperature. These effects have repeatedly been observed for gas flows in tubes [6,31,56].

On this basis we assume that crisis can occur in channels of fixed cross-section subject to heat influx.

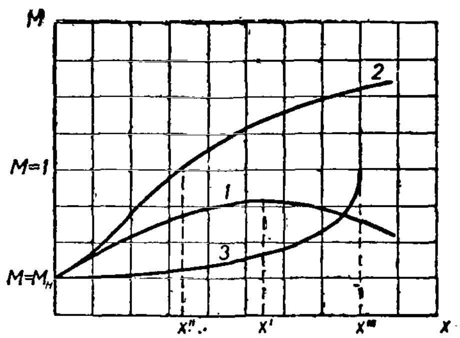

Fig. 1.

Variation in M along the channel for various types of over-all action: 1) speed nowhere equals or exceeds that of sound, over-all action changes sign at $x = x'$; 2) speed attains local speed of sound at $x = x''$, where over-all action changes sign; 3) over-all action always positive, crisis occurs at $x = x'''$.

We now consider some particular cases.

3. Adiabatic Flows

Subsonic-Supersonic Transition and Condition for Crisis

An adiabatic flow implies no heat transfer between gas and wall ($d\Theta/dx = 0$).

(1.48) implies that a subsonic flow will accelerate if

$$\frac{1}{F}\frac{dF}{dx} < \zeta \frac{kM^2}{2D}$$

so the channel may narrow or may widen gradually.

The point at which the speed of sound is reached is of interest; (1.49) gives that at this point

$$\frac{D}{F}\frac{dF}{dx} = \frac{k\zeta}{2}.$$

The local angle of divergence of the channel [9] is

$$\tan\frac{\varphi}{2} = \frac{1}{S_w}\frac{dF}{dx},$$

in which S_w is the wetted perimeter.

By definition, the hydraulic diameter is

$$D = \frac{4F}{S_w}$$

so

$$\tan\frac{\varphi}{2} = \frac{D}{4F}\,\frac{dF}{dx},$$

and at the point of transition we must have

$$\tan\frac{\varphi}{2} = \frac{k\zeta}{8}\,.$$

ζ is usually fairly small, so the transition occurs in a part of the channel where the widening is rather gradual.

The condition for onset of crisis in a subsonic flow is as follows. The condition (section 2) is that we have at some finite distance

$$\zeta\,\frac{kM^2}{2D} - \frac{1}{F}\,\frac{dF}{dx} > 0$$

for M close to one, including M = 1.

Flow in a Channel of Constant Cross-Section

In this case $dF/dx = 0$, so the above inequality is bound to apply (since $\zeta > 0$), and crisis can occur.

We put in (1.29) that

$$\frac{dF}{dx} = \frac{d\Theta}{dx} = 0, \; \zeta = \text{constant},$$

and after integration get

$$\chi\,(M_{*1}) - \chi\,(M_*) = \zeta\,\frac{x}{D}, \qquad (1.50)$$

in which

$$\chi\,(M_*) = \frac{k+1}{2k}\left(\frac{1}{M^2} + \ln M_*^2\right). \qquad (1.51)$$

Flow with M = constant

In this case Θ = constant and M = constant together; (1.23) shows that we can have M = constant if

$$\frac{D}{F}\,\frac{dF}{dx} = \zeta\,\frac{kM^2}{2},$$

or, in terms of the local angle of taper (widening),

$$\tan\frac{\varphi}{2} = \frac{1}{8}\,\zeta k M^2,$$

so the channel must widen slowly for such a flow.

4. Flows with Heat Transfer

Flow with M = constant and F = constant

Such a flow can occur [28]; (1.23) readily shows that

$$\frac{1}{\Theta}\frac{d\Theta}{dx} = -\frac{\zeta}{D}\frac{kM^2}{1 + kM^2}$$

so with ζ = constant we have

$$\Theta = \Theta_i \exp\left(-\frac{\zeta}{D}\frac{kM^2}{1 + kM^2}x\right),$$

so $\Theta \to 0$ for $x \to \infty$.

Now (1.15) gives T/Θ = constant, so $T \to 0$ and hence, from (1.14), $w \to 0$. In this case the flow speed and the limiting speed both approach zero.

Isothermal Flow in a Channel of Constant Cross-Section

Here $dT/dx = dF/dx = 0$, so from (1.43) we have

$$(1 - kM^2)\frac{2 + (k - 1)M^2}{2\Theta}\frac{d\Theta}{dx} = \frac{k - 1}{2}\zeta\frac{kM^4}{D}.$$

Then for $M^2 < 1/k$ the gas must be heated to keep the flow isothermal ($d\Theta/dx > 0$); (1.23) indicates that M then increases, and the heat influx must increase with M, with finally $d\Theta/dx \to \infty$ as

$M^2 \to 1/k$, so $q \to \infty$.

This means that an isothermal flow with F = constant is possible only for $M^2 < 1/k$ [25,27].

Flow in a Channel of Constant Cross-Section without Loss

This case ($\zeta = 0$) is of some theoretical interest; (1.29) is then readily integrated to give

$$\left(\frac{\Theta_f}{\Theta_i}\right)_{\zeta=0} = \left(\frac{M_{*f}}{M_{*i}}\right)^2 \left(\frac{1 + M_{*i}^2}{1 + M_{*f}^2}\right)^2 . \tag{1.52}$$

Θ_f/Θ_i is the degree of heating (degree of increase in the retardation temperature); (1.52) shows that it is dependent only on M_{*i} and M_{*f} for $\zeta = 0$, so M_{*f} (exit value) is uniquely determined by M_{*i} (input value) and by Θ_f/Θ_i no matter what the mode of influx of heat.

Flow in a Channel of Constant Cross-Section with Loss, and Influx of Heat at Inlet and Outlet

The effects become more complicated when frictional loss occurs, partly because M_{*f} is now dependent on ζ as well as on M_{*i} and Θ_f/Θ_i, and partly because M_{*f} is dependent on the mode of influx, as (1.29) shows. The effect of the latter may be evaluated by reference to two limiting cases: influx of heat at inlet and outlet.

We assume that the channel is extended for some distance beyond the inlet and outlet sections, but in these idealized parts there is no friction. We say that there is heat influx at the inlet if all the heat enters in the idealized section preceding the inlet; and similarly for the outlet.

Heat influx at the inlet causes M_* to rise from M_{*i} to M'_{*i}, the latter being related to the degree of heating and to M_{*i} by (1.52):

$$\frac{\Theta_f}{\Theta_i} = \left(\frac{M'_{*i}}{M_{*i}}\right)^2 \left(\frac{1 + M^2_{*i}}{1 + M'^2_{*i}}\right)^2 . \tag{1.53}$$

The subsequent flow involves no heat transfer, so from (1.50)

$$\chi\,(M'_{*i}) - \chi\,(M_{*f}) = \zeta\bar{L}, \tag{1.54}$$

in which $\bar{L}$ is the length of the channel referred to the hydraulic diameter.

Heat influx at the outlet causes the speed to rise adiabatically from M_{*i} to M'_{*f} in accordance with

$$\chi\,(M_{*i}) - \chi\,(M'_{*f}) = \zeta\bar{L}. \tag{1.55}$$

Further increase in speed results solely from the influx of heat, for which

$$\frac{\Theta_f}{\Theta_i} = \left(\frac{M_{*f}}{M_{*i}}\right)^2 \left(\frac{1 + M'^2_{*f}}{1 + M^2_{*f}}\right)^2 \tag{1.56}$$

Figure 2 shows Θ_f/Θ_i as a function of M_{*f} for the above two cases for $M_{*i} = 0.2$ and $\zeta\bar{L} = 1$. The difference between the two is considerable, and it increases with M_{*f}.

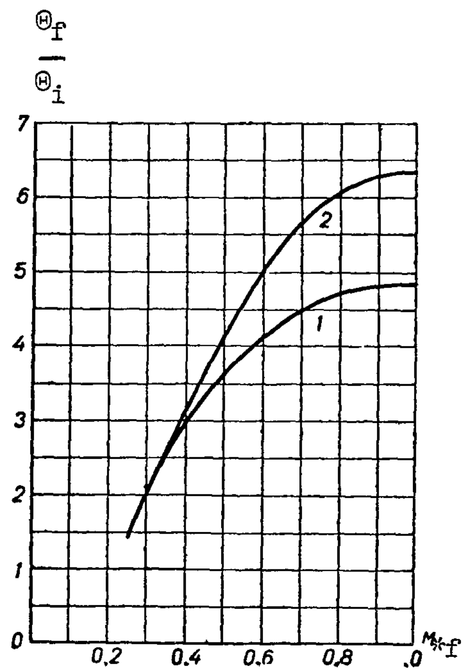

Fig. 2.

Relation of Θ_f/Θ_i for a channel of constant cross-section to M_{*f} for $M_{*i} = 0.2$ and $\zeta\bar{L} = 1$; heat influx at: 1) inlet; 2) outlet.

Figure 2 shows that influx at the inlet may give M_{*f} much greater than that for influx at the outlet for the given $\zeta\bar{L}$ and M_{*i}.

For instance, for $\Theta_f/\Theta_i = 4.85$ we have M_{*f} as unity in the first case but less than 0.6 in the second; the reason is as follows. In the first case the friction begins to have an effect at the fairly high M_* resulting from the heating, so, even if it is small, it can raise M_* to one. In the second case the friction acts while M_* is small and so gives no substantial rise in M_*; hence heating cannot give the results obtained in the first case.

Flow in a Channel of Constant Cross-Section, Heat Influx along Entire Length

A simplified relation of the degree of heating to M_{*i}, M_{*f}, and ζ is needed for some general purposes, such as those dealt with in section 5. This relation should correspond to some average type of heat influx intermediate between the above two cases.

This relation is readily derived as follows.

From (1.29) with $dF/dx = 0$ we have

$$\frac{\Theta_f}{\Theta_i} = \left(\frac{\Theta_f}{\Theta_i}\right)_{\zeta=0} \exp\left(-\frac{2k}{k+1}\zeta\bar{L}\int_0^1 \frac{M_*^2}{1+M_*^2}\, d\frac{x}{L}\right). \qquad (1.57)$$

Here the preexponential factor is the degree of heating found from (1.29) for $\zeta = 0$ for the same M_i and M_k.

An approximate value for the integral may [16] be found as follows.

We have from (1.29) with $\zeta = 0$ and $dF/dx = 0$ that

$$\frac{1 - M_*^2}{M_*^3}\frac{dM_*^2}{dx} = \frac{1 + M_*^2}{M_*^2}\frac{d\ \ln\Theta}{dx},$$

or

$$-\frac{2k}{k+1}\frac{d\chi\,(M_*)}{d\frac{x}{L}} = \frac{1+M_*^2}{M_*^3}\frac{d\ \ln\Theta}{d\frac{x}{L}},$$

in which $\chi(M_*)$ is given by (1.51). Then

$$\ln\left(\frac{\Theta_f}{\Theta_i}\right)_{\zeta=0} = -\frac{2k}{k+1}\int_0^1 \frac{M_*^2}{1+M_*^2}\frac{d\chi(M_*)}{d\frac{x}{L}}\,d\frac{x}{L}\ .$$

We extract the mean value of $d\chi/d(x/L)$ from the integral and replace it by $\chi(M_{*f}) - \chi(M_{*i})$ to get

$$\frac{2k}{k+1}\int_0^1 \frac{M_*^2}{1+M_*^2}\,d\frac{x}{L} = \frac{\ln\left(\frac{\Theta_f}{\Theta_i}\right)_{\zeta=0}}{\chi(M_{*i}) - \chi(M_{*f})}$$

This is substituted into (1.57) to give

$$\frac{\Theta_f}{\Theta_i} = \left(\frac{\Theta_f}{\Theta_i}\right)_{\zeta=0}^{1-\frac{\zeta\bar{L}}{\chi(M_{*i})-\chi(M_{*f})}} \qquad (1.58)$$

This reduces to known relationships in two limiting cases: when $\Theta_f/\Theta_i = 1$ (no heat transfer) it becomes (1.50), and when $\zeta = 0$ (no friction) it becomes (1.52).

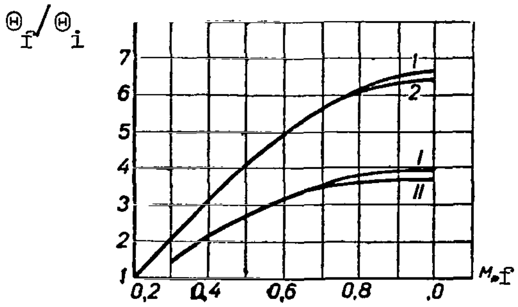

Fig. 3.

Comparison of degrees of heating for $M_{*i} = 0.2$: 1) and I) from (1.58); 2) and II) half the sum of (1.53) and (1.56). $\zeta\bar{L}$ as follows: 1) and 2) 0.25; I) and II) 5.

Figure 3 shows how Θ_f/Θ_i as given by (1.58) differs from half the sum of the values given by the above two cases; the values are for $M_{*i} = 0.2$.

The agreement is good, so (1.58) gives Θ_f/Θ_i intermediate between the values for influx at inlet and outlet.

Flow in a Widening Channel for M = constant

(1.23) with M = constant gives us the relation between rate of increase of area, temperature, and resistance:

$$\frac{1}{F}\frac{dF}{dx} = \frac{1 + kM^2}{2\Theta}\frac{d\Theta}{dx} + \zeta\frac{kM^2}{2D} \qquad (1.59)$$

A channel of reasonable length should not diverge rapidly for reasonably small M and not very large degrees of heating.

The local angle of divergence φ may be used to characterize the widening (section 3); this is given by

$$\tan\frac{\varphi}{2} = \frac{D}{4F}\frac{dF}{dx}. \qquad (1.60)$$

(1.59) with M sufficiently small gives

$$\frac{1}{F}\frac{dF}{dx} \approx \frac{1}{2\Theta}\frac{d\Theta}{dx}, \qquad (1.61)$$

so

$$\tan\frac{\varphi_m}{2} = \frac{1}{L}\int_0^L \tan\frac{\varphi}{2}\,dx \cong \frac{D_m}{8L}\ln\frac{\Theta_f}{\Theta_i}. \qquad (1.62)$$

Here φ_m is the mean value of φ; D_m is a value of D intermediate between those for the inlet and outlet.

This relation shows that for $\Theta_f/\Theta_i < 4$ and $L/D_m > 50$ we have $\varphi_m = 0.5°$, which means that the resistance coefficient will be almost exactly that for channels of constant cross-section [9].

5. Maximal Degree of Heating and Efficiency of Supply of Heat to a Channel of Constant Cross-Section

We have seen (section 2) that $M \leq 1$ when heat flows into a channel of constant cross-section*, so the degree of heating is also restricted. The greatest degree of heating for $\zeta = 0$ is found by putting $M_{*f} = 1$ in (1.52):

$$\left(\frac{\Theta_f}{\Theta_i}\right)^{max}_{\zeta=0} = \left(\frac{1 + M^2_{*i}}{2M_{*i}}\right)^2 \tag{1.63}$$

(1.58) gives us for friction present that

$$\left(\frac{\Theta_f}{\Theta_i}\right)^{max} = \left[\left(\frac{\Theta_f}{\Theta_i}\right)^{max}_{\zeta=0}\right]^{1 - \frac{\zeta\bar{L}}{\chi(M_{*i}) - \chi(1)}} \tag{1.64}$$

Figure 4 shows results for this as found from (1.63) and (1.64).

High degrees of heating are obtainable for fairly large $\zeta\bar{L}$ only if M_{*i} is small. For instance, $(\Theta_f/\Theta_i)^{max} > 1.5$ is obtainable with $\zeta\bar{L} = 5$ only for $M_{*i} < 0.3$. This requires us to consider possible ways of reducing M_{*i}.

The referred flow rate $g(M_*)$ increases with M_* for $M_* < 1$, so

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*(1.28) also implies that $M \leq 1$ in this case.

for fixed F and Θ_i we can reduce M_{*i} only either by decreasing the flow rate G or by increasing the pressure P_i, as (1.37) shows.

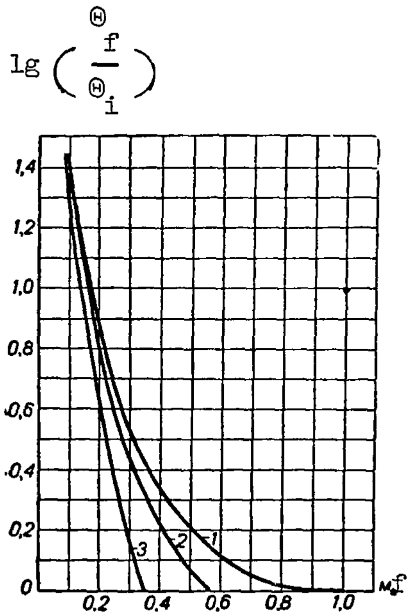


Fig. 4.

Relation of $\log(\Theta_f/\Theta_i)^{max}$ to M_{*i} for $\zeta\bar{L}$ of: 1) 0; 2) 1; 3) 5.

It is undesirable to reduce the flow rate if the gas is to be used in an apparatus that cannot work efficiently with a flow less than a certain value. In addition, reduction of the flow rate with a given degree of heating reduces the amount of heat removed per second by the gas; so reduction in M_{*i} usually requires a rise in P_i, and hence the maximum degree of heating is restricted by the permissible pressure.

We shall now see that reduction in M_{*i} not only enables us to increase the maximum degree of heating but also raises the efficiency of supply of heat to the gas. This efficiency is characterized by $\varkappa$, the ratio of the amount of heat Q carried off each second by the gas to the power N required to pump the gas:

$$\varkappa = \frac{Q}{N}.$$

The efficiency clearly increases with $\varkappa$. The heat carried off each second by the gas is

$$Q = Gc_p\Theta_i\left(\frac{\Theta_f}{\Theta_i} - 1\right). \tag{1.65}$$

N is found as the work needed for isentropic compression of the gas from the pressure p_f at the outlet to the pressure P_i at the inlet; it is given by

$$N = Gc_p\Theta_i\left(1 - \sigma^{\frac{k-1}{k}}\right) \tag{1.66}$$

in which

$$\sigma = \frac{p_f}{P_i}$$

This σ is termed the pressure coefficient. It is assumed in (1.66) that the kinetic energy the gas has at the outlet has been completely lost, which is not always the case; but this assumption is quite justified for the purposes of qualitative analysis.

(1.62) and (1.66) enable us to replace (1.65) by

$$x = \frac{\dfrac{\Theta_f}{\Theta_i} - 1}{1 - \sigma^{\frac{k-1}{k}}} . \qquad (1.67)$$

(1.37) and (1.39) can also be used to find σ; the first is written for the initial section and the second for the exit, the one being divided by the other to give

$$\sigma = \frac{F_i}{F_f} \sqrt{\frac{\Theta_f}{\Theta_i} \frac{g(M_{*i})}{Y(M_{*f})}} . \qquad (1.68)$$

(1.68) applies to any channel, not merely one of constant cross-section. In the present case (constant cross-section) we put $F_i = F_f$ in (1.68).

This can be used with various values of M_{*f} in the range $[M^a_{*f}, 1]$, in which M^a_{*f} is M_* for adiabatic flow (no heating), which is given by

$$\chi\,(M_{*i}) - \chi\,(M^a_{*f}) = \zeta\bar{L}.$$

Taking M_{*i} and $\zeta\bar{L}$ as fixed, we find Θ_f/Θ_i from (1.58), σ from (1.68), and $\varkappa$ from (1.65). Then each Θ_f/Θ_i corresponds to a certain $\varkappa$ for given M_{*i} and $\zeta\bar{L}$, so $\varkappa$ is a function of Θ_f/Θ_i with M_{*i} and $\zeta\bar{L}$ as parameters.

There is a maximum in $\varkappa$, whose height is dependent on $\zeta\bar{L}$ and M_{*i}; this is the largest for a given M_{*i} when $\zeta = 0$ (no loss) and is termed $\varkappa_{max}$.

Figure 5 shows $\varkappa_{max}$ as a function of M_{*i}, which to some extent indicates the effects of increased pressure on the efficiency of

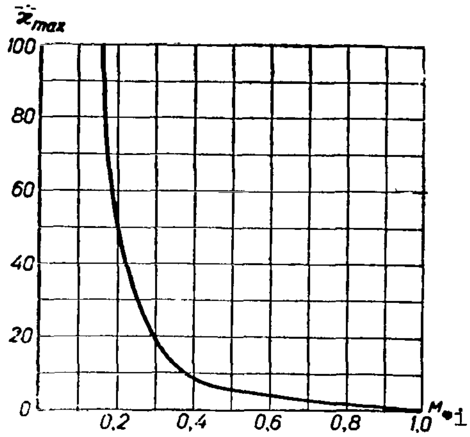


Fig. 5.

Relation of $\varkappa_{max}$ to M_{*i}.

heat transfer. Reduction of M_{*i} from 0.3 to 0.15 (which corresponds to increasing P_i by roughly a factor two at fixed flow rate) causes $\varkappa$ to increase by over a factor five.

Figures 6 and 7 give a reasonably complete indication of the effects of Θ_f/Θ_i and increased pressure on $\varkappa$; they show as ordinate $\varkappa$ referred to the $\varkappa_{max}$ for given M_{*i}.

Figures 6 and 7 clearly illustrate the effects of $\zeta\bar{L}$ on $\varkappa$ for given M_{*i}.

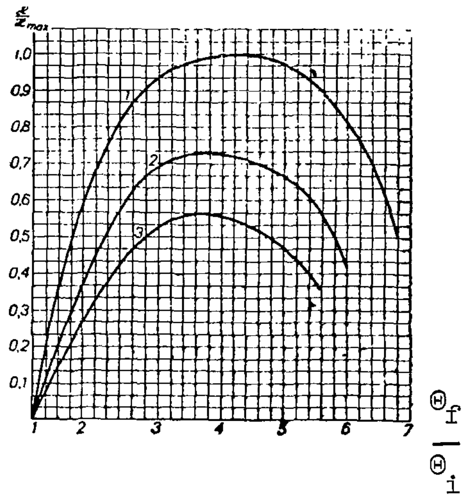
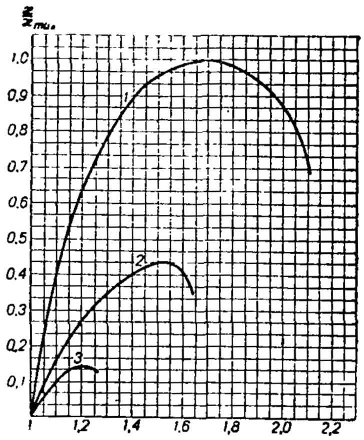


Fig. 6.

Relation of $\varkappa$ to Θ_f/Θ_i and ζ for M_{*i} = 0.2 and $\zeta\bar{L}$ of: 1) 0; 2) 1; 3) 2.

Figure 8 shows $\varkappa$ as a function of Θ_f/Θ_i for M_{*i} of 0.2 and 0.4 ($\zeta\bar{L}$ = 1). It is clear that $\varkappa$ increases appreciably when M_{*i} is decreased for a given Θ_f/Θ_i.

For example, reduction of M_{*i} from 0.4 to 0.2 for Θ_f/Θ_i = 1.5 causes $\varkappa$ to increase from 3.44 to 10 (by a factor 2.9). The effect is more pronounced if M_{*i} is reduced (e.g., from 0.4 to 0.2) while Θ_f/Θ_i is increased (e.g., from 1.5 to 2.4); this causes $\varkappa$ to increase from 3.44 to 21.5 (by a factor 6.25).



$\frac{\Theta_f}{\Theta_i}$

Fig. 7.

Relation of $\varkappa$ to Θ_f/Θ_i and ζ for $M_{*i} = 0.4$ and $\zeta\bar{L}$ of: 1) 0; 2) 1; 3) 2.

For any given M_{*i} and specified $\zeta\bar{L}$ there is an optimal Θ_f/Θ_i (that giving the largest $\varkappa$). For example, for $M_{*i} = 0.2$ and $\zeta\bar{L} = 1$ (Fig. 6) this optimal value is close to 4, while for $M_{*i} = 0.4$ and $\zeta\bar{L} = 1$ (Fig. 7) it is close to 1.5.

A knowledge of the optimal Θ_f/Θ_i enables one to select the most efficient conditions of operation.

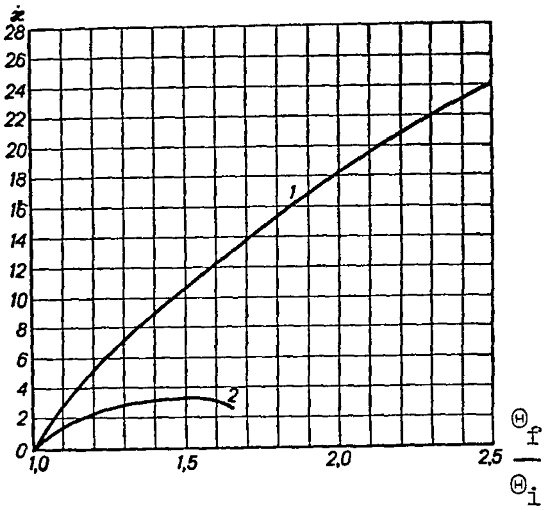


Fig. 8.

Effects of M_{1i} and Θ_f/Θ_i on $\varkappa$ for M_{*i} of: 1) 0.2; 2) 0.4.

6. Comparison of Efficiency of Supply of Heat in Various Channels

A real channel is frequently a part of a closed loop (as in a nuclear reactor); the gas passes from the exit via various pipes back to the inlet. The power consumed in circulating the gas will then be dependent on the resistance of these other pipes as well. The power is given by

$$\frac{P_f'}{P_i} = \frac{p_f}{P_i}\,\frac{P_f'}{p_f} = \sigma\,\frac{P_f'}{p_f}$$

in which P_f'/P_f specifies resistance of the pipes. This means that channels under consideration for use in a given plant are best compared by reference to the σ needed to take up a fixed quantity Q of heat. The best channel has the largest σ.

We assume that Θ_i, p_f, and G (gas flow rate) are constant.

Then $G^{I} = G^{II}$, $Q^{I} = Q^{II}$, $\Theta_i^{I} = \Theta_i^{II}$, so (1.65) gives $(\Theta_f/\Theta_i)^{I} = (\Theta_f/\Theta_i)^{II} = \Theta_f/\Theta_i$, so $\Theta_f^{I} = \Theta_f^{II} = \Theta_f$.

The channels then produce the same degree of heating of the gas. But $p_f^{I} = p_f^{II}$, so (1.39) gives

$$F_f^{I} Y(M_{*f}^{I}) = F_f^{II} Y(M_{*f}^{II}). \tag{1.69}$$

Then (1.68) and (1.69) give

$$\frac{\sigma^{I}}{\sigma^{II}} = \frac{F_i^{I}}{F_i^{II}} \frac{g(M_{*i}^{I})}{g(M_{*i}^{II})} \tag{1.70}$$

This enables us to compare channels for efficiency in a fairly simple way. Two cases are considered below to illustrate this.

Comparison of Efficiencies for Heat Influx at Inlet and Outlet for a Channel of Constant Cross-Section

Consider a channel of fixed cross-section which in one case receives heat at the outlet (channel I) and in the other at the inlet (channel II). Now $F_f^{I} = F_f^{II}$, so (1.69) gives $M_{*f}^{I} = M_{*f}^{II} = M_{*f}$.

We consider for simplicity the limiting case $M_{*f} = 1$; Fig. 9 gives results from (1.53)-(1.56) and (1.70).

Influx of heat at the outlet is more favorable than that at the inlet.

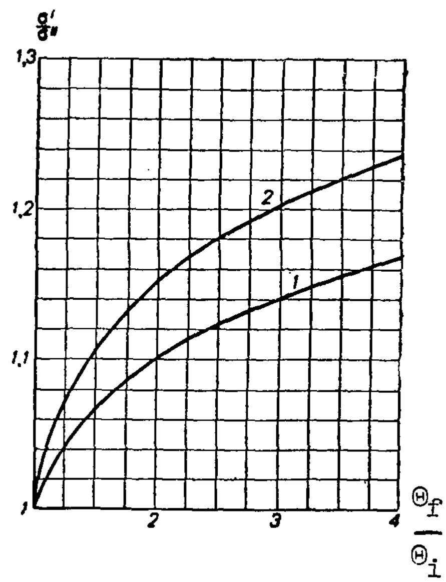


Fig. 9.

Comparison of efficiencies for heat influx at inlet and outlet for a channel with F = constant for $\zeta\bar{L}$ of: 1) 1; 2) 2.

Section 4 shows that influx at outlet or inlet cannot actually occur in pure form; but the results obtained here are still of practical interest. The greater the heat influx at the start of the channel, the closer the case to that of influx at the inlet; and similarly for the outlet. The analysis indicates that the efficiency of a channel may be raised by bringing it as close as possible to case I (not case II); as much as possible of the heat Q should be supplied as close as possible to the outlet. Figure 9 shows

σ^I/σ^{II}, the maximum possible (theoretical) gain.

For instance, for a maximal Θ_f/Θ_i of three and $\zeta\bar{L} = 2$ we have $\sigma^I/\sigma^{II} \approx 1.2$; $p_f^I = p_f^{II}$, so $P_f^{II} \approx 1.2P_f^I$. Then a threefold rise in retardation temperature of the gas for $\zeta\bar{L}$ can be produced in channel I with an initial retardation pressure 20% lower than that for channel II.

Comparison of Channels with M = constant and F = constant

Channel I has M = constant; channel II, F = constant.

Let channel II have $\zeta\bar{L} = 1$, and $M_{*i}^{II} = 0.2$, and $\Theta_f/\Theta_i = 4$, which is close to the optimal value (Fig. 7).

Channel I has the same inlet cross-section as II: $F_i = F_i^{II} = F_f^{II}$.

(1.58) with the given Θ_f/Θ_i, $\zeta\bar{L}$, and M_{*i}^{II} gives $M_{*f}^{II} = 0.5$.

(1.29) gives, after integration and use of $M_* =$ constant, that

$$\left(\frac{F_f^I}{F_i^I}\right)^{1 - \frac{k-1}{k+1}M_*^2} = \left(\frac{\Theta_f}{\Theta_i}\right)^{\frac{1 + M_*^2}{2}} e^{\zeta \frac{k}{k+1} M_*^2 \frac{L}{D_m}}$$

in which

$$D_i < D_m < D_f.$$

From this and (1.69) we have

$$M_{*_i}^{I} = M_{*_f}^{I} = M_{*}^{I} = 0.23,$$

so from (1.70)

$$\frac{\sigma^{I}}{\sigma^{II}} = \frac{g(M_{*_i}^{I})}{g(M_{*_i}^{II})} \approx 1.15.$$

The channel with M = constant is better than that with F = = constant; this is attained by increasing the cross-sectional area by a factor 2.25, which corresponds to an increase in diameter by a factor 1.5.

The above analysis is not complete, of course; it must be refined for each particular case by reference to experimental evidence on the coefficients of heat transfer and friction as functions of channel geometry, method of supplying heat, temperature, and perhaps gas pressure.

CHAPTER II

GAS FLOW IN A SYMMETRICALLY HEATED CHANNEL WITH PLANE-PARALLEL WALLS: METHOD OF CALCULATION

The one-dimensional theory of the previous chapter is based on the assumption that the parameters (speed, temperature, density, pressure) are independent of the transverse coordinate, being functions of the lengthwise coordinate alone. It has been shown [39] that this is almost so for flows in sufficiently long tubes at high values of Reynolds number.

On the other hand, there are several problems that the one-dimensional theory cannot solve, including that of the character of the flow near the crisis point.

Further, the resistance coefficient ζ and the heat-transfer factor α of the one-dimensional theory can be given a theoretical basis only from a more detailed study of the flow in channels. For this we need to use the equations for a viscous compressible fluid, which are very complicated even in the limiting case Re $\to \infty$ (boundary-layer equations).

There is an extensive literature on boundary layers and heat transfer [17-19, 21, 23, 32, 34, 36-38, 42, etc.]

Here I consider the simplest case, namely symmetrical flow in a plane-parallel channel.

We shall see that the methods developed for plane-parallel channels can be applied also to circular tubes.

Calculations on the boundary layer in a channel involve the need to solve the interior problem, which differs from the exterior one (in which the speed of the gas at the boundary is a known function of the lengthwise coordinate) in that the speed at the axis is now one of the unknowns.

Laminar adiabatic flow of an incompressible fluid has known solutions for channels with plane-parallel walls [58] and for circular pipes [38, 57]; there are also solutions for the adiabatic flow of a gas in the laminar [11] and turbulent [1, 41] modes.

There are papers on laminar flows of incompressible fluid in pipes in the presence of heat transfer [46, 47, 53, etc.], and also many on turbulent flow in pipes [18, 23, 44, 48, 50, etc.]. These have as object, in particular, the derivation of the effects of Pr on the relation between the heat-transfer and resistance coefficients (i.e., on revision of the Reynolds analogy for $Pr \neq 1$).

There have recently appeared several fresh studies of turbulent gas flow in pipes in the presence of heat transfer, in particular [20, 41].

Some simplifying assumptions were made in [41]; in particular, it was assumed that the density and viscosity are independent of the temperature and of the transverse coordinate.

The theory of local similitude [12, 13] was used in [20] to analyze experimental data; relationships were given for the heat transfer in the initial part of the tube.

Here I give a new approximate method mainly directed to two purposes: definition of the limits of application of the Reynolds analogy and a more detailed study of the effects near crisis.

In principle, the method is applicable for all Pr, but I assume that $Pr = 1$, which greatly simplifies the equations for the boundary layer and which much reduces the volume of analysis and of calculation.

This limitation of the purpose leads me to put $Pr = 1$ and to study the conditions for deviation from Reynolds analogy associated with factors other than the deviation of Pr from one, especially since the effects of Pr on that analogy are already largely known. I assume that ϵ (turbulent-transfer factor for momentum) and ϵ' (the same for heat) are the same: $\epsilon = \epsilon'$.

In addition, I assume that $Re < 10^5$, although analogous results can be derived* for $Re > 10^5$. The assumption $Re < 10^5$ enables me to simplify the treatment, because the velocity profile in this range is adequately described by a power law (the 1/7 law), so the problem reduces to that of examining the effects of M and heat transfer.

1. Basic Equations

Consider the steady-state turbulent flow of a gas in a channel

*See Section 13 of this chapter.

of width 2h. The Ox axis lies along one wall and is directed downstream; the Oy axis is perpendicular to the axis of the channel and is directed into the gas (Fig. 10).

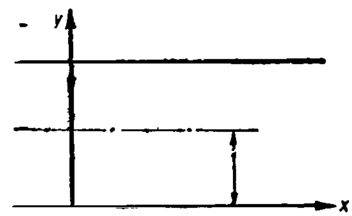


Fig. 10.

Plane-parallel channel.

The speed at the axis in the initial section is assumed subsonic. The wall temperature or specific heat flux can be any function of x in the general case. Here I assume that the two functions are identical for the two walls (symmetrical case).

The set of equations for the boundary layer in terms of time-averages takes the form

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{dp}{dx} + \frac{\partial}{\partial y} \left[(\mu + \epsilon) \frac{\partial u}{\partial y} \right], \qquad (2.1)$$

$$\rho \left(u \frac{\partial \Theta}{\partial x} + v \frac{\partial \Theta}{\partial y} \right) = \frac{\partial}{\partial y} \left[(\mu + \epsilon) \frac{\partial \Theta}{\partial y} \right] \qquad (2.2)$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0, \qquad (2.3)$$

$$p = \rho RT, \qquad (2.4)$$

$$\Theta = T + \frac{k-1}{2k} \cdot \frac{u^2}{R}. \qquad (2.5)$$

System (2.1)-(2.5) is the same for the exterior problem as for the interior one.

dp/dx is known for the former, but the pressure is one of the unknowns for the latter, though we do have here an additional conditions (constancy of the flow rate along the channel).

We put (2.1)-(2.5) in dimensionless form for convenience in further exposition.

We take half the width of the channel as scale for the transverse coordinate y: $y = \bar{y}h$. The symmetry allows us to discuss only half the width in what follows.

Values at the wall ($\bar{y} = 1$) by subscript 1.

The scale for the velocity components u and v is the limiting velocity at the center of the initial section of the channel:

$$u = \bar{u}u^{(1)}_{1i} \qquad v = \bar{v}u^{(1)}_{1i} .$$

We also put

$$p = \bar{p}P_i, \quad T = \bar{T}\Theta_{1i} \quad \rho = \bar{\rho}\,\frac{P_i}{R\Theta_{1i}},$$

$$\Theta = \bar{\Theta}\Theta_{1i}, \quad \mu = \bar{\mu}\mu'_{1i}, \quad \epsilon = \bar{\epsilon}\mu'_{1i}$$

in which μ'_{1i} is the viscosity at the axis in the initial section for the stagnation temperature Θ_{1i}.

The Re defined from $u^{(1)}_{1i}$ and μ'_{1i} is denoted by $\widetilde{Re}$:

$$\widetilde{Re} = \frac{u_{1i}^{(1)} h P_i}{\mu'_{1i} R \Theta_{1i}} \tag{2.6}$$

and we put

$$x = \bar{x}\, h\, \widetilde{Re}. \tag{2.7}$$

Simple manipulations based on (1.45) enable us to put (2.1)-(2.5) in the following form (the bars over the dimensionless quantities are here and henceforth omitted)*:

$$\rho\left(u\,\frac{\partial u}{\partial x} + \widetilde{Re}\, v\,\frac{\partial u}{\partial y}\right) = -\,\frac{k-1}{2k}\,\frac{dp}{dx} + \frac{\partial}{\partial y}\left[(\mu + \epsilon)\frac{\partial u}{\partial y}\right], \tag{2.8}$$

$$\rho\left(u\,\frac{\partial \Theta}{\partial x} + \widetilde{Re}\, v\,\frac{\partial \Theta}{\partial y}\right) = \frac{\partial}{\partial y}\left[(\mu + \epsilon)\,\frac{\partial \Theta}{\partial y}\right], \tag{2.9}$$

$$\frac{\partial(\rho u)}{\partial x} + \widetilde{Re}\,\frac{\partial(\rho v)}{\partial y} = 0, \tag{2.10}$$

$$p = \rho T, \tag{2.11}$$

$$\Theta = T + u^2. \tag{2.12}$$

---

*Special mention is made in what follows of any operation involving dimensional quantities.

2. Boundary Conditions

The system (2.8)-(2.12) must be complemented with boundary conditions.

The condition for the velocity is that of adherence to the wall:

$$u_0 = v_0 = 0. \tag{2.13}$$

The condition for stagnation temperature Θ varies in accordance with whether the wall temperature or the heat flux to the gas is the known quantity.

In the first case, clearly, Pr = 1:

$$\Theta_0 = T_0 = T_0(x), \tag{2.14}$$

in which $T_0(x)$ is wall temperature*.

In the second case we take account of the variation in thermal conductivity with temperature and put that

$$\left(\frac{\partial T}{\partial y}\right)_0 = -\frac{q_0(x)}{\lambda_0}.$$

Here $q_0(x)$ is a known function of x (the heat flux at the wall) and

*The wall temperature is denoted hereinafter by Θ_0 for convenience.

λ_0 is a function of T_0.

Now (2.12) gives

$$\left(\frac{\partial\Theta}{\partial y}\right)_0 = \left(\frac{\partial T}{\partial y}\right)_0 + 2u_0\left(\frac{\partial u}{\partial y}\right)_0,$$

so (2.13) gives

$$\left(\frac{\partial\Theta}{\partial y}\right)_0 = \left(\frac{\partial T}{\partial y}\right)_0,$$

and the condition at the wall may be put as

$$\left(\frac{\partial\Theta}{\partial y}\right)_0 = -\frac{q_0(x)}{\lambda_0}. \tag{2.15}$$

3. Integral Relations

An approximate method will be used to solve (2.8)-(2.12), so integral relations derived from these equations are deduced.

We integrate (2.10) with respect to y between wall and axis (i.e., from $y = 0$ to $y = 1$) to get

$$\frac{d}{dx}\int_0^1 \rho u dy + \widetilde{Re}\,[\rho v]_0^1 = 0.$$

(2.13) gives $v_0 = 0$, and $v_1 = 0$ by virtue of the symmetry, so

$[\rho v]_0^1 = 0$ and hence

$$\frac{d}{dx} \int_0^1 \rho u dy = 0. \tag{2.16}$$

The integral is the mass of gas flowing each second through any cross-section. (2.16) implies that this mass (the mass flow rate) is the same for all points. Let G denote the flow rate (as in chapter I); then instead of (2.16) we have*

$$\int_0^1 \rho u dy = G_1 = \text{const.} \tag{2.17}$$

We integrate (2.8) with respect to y = 0 to y = 1 to get, taking all terms to the left, that

$$\int_0^1 \rho u \frac{\partial u}{\partial x} dy + \widetilde{Re} \int_0^1 \rho v \frac{\partial u}{\partial y} dy + \frac{k - 1}{2k} \frac{dp}{dx} - \left[(\mu + \epsilon)\frac{\partial u}{\partial y}\right]_0^1 = 0. \tag{2.18}$$

The second integral is taken by parts and (2.10) is used to give

$$\widetilde{Re} \int_0^1 \rho v \frac{\partial u}{\partial y} dy = \widetilde{Re}[\rho v u]_0^1 - \widetilde{Re} \int_0^1 u \frac{\partial(\rho v)}{\partial y} dy = \int_0^1 u \frac{\partial(\rho v)}{\partial x} dy,$$

---

*G_1 here an henceforth denotes the flow rate for half the width of the channel.

so the sum of the first two terms in (2.18) is

$$\int_0^1 \rho u \frac{\partial u}{\partial x}\, dy + \widetilde{Re} \int_0^1 \rho v \frac{\partial u}{\partial y}\, dy = \frac{d}{dx}\int_0^1 \rho u^2 dy.$$

$(\partial u/\partial y)_1 = 0$, on account of the symmetry, while $\epsilon_0 = 0$ (turbulent transfer coefficient at the wall), so (2.18) is replaced by

$$\frac{d}{dx}\int_0^1 \rho u^2 dy + \frac{k-1}{2k}\frac{dp}{dx} + \mu_0 \left(\frac{\partial u}{\partial y}\right) = 0. \qquad (2.19)$$

Entirely analogously we have from (2.9) and (2.10)

$$\frac{d}{dx}\int_0^1 \rho u \Theta dy + \mu_0 \left(\frac{\partial \Theta}{\partial y}\right)_0 = 0. \qquad (2.20)$$

Note that (2.19) and (2.20) are extensions of (1.7) and (1.8) to the case of nonuniform velocity and temperature distributions over the channel cross-section.

Also

$$\frac{1}{G_1}\int_0^1 \rho u \Theta dy = \Theta_m \qquad (2.21)$$

is the mean-mass stagnation temperature, while

$$\frac{1}{G_1}\int_0^1 \rho u^2 dy = u_m \tag{2.22}$$

is the mean-mass speed.

(2.20) and (2.21) together can be put as

$$G_1 \frac{d\Theta_m}{dx} = -\mu_0 \left(\frac{\partial\Theta}{\partial y}\right),$$

or, from (2.15)

$$G_1 \frac{d\Theta_m}{dx} = \frac{\mu_0}{\lambda_0} q_0. \tag{2.23}$$

Pr = 1 gives (with all quantities except Pr dimensional)

$$Pr = \frac{c_p \mu_0}{\lambda_0} = 1.$$

Also, $S_h = 2$, so reduction of (1.7) to dimensionless form readily enables us to show that it is the same as (2.23).

Now we put (2.19) in the following form by means of (2.22):

$$G_1 \frac{du_m}{dx} + \frac{k - 1}{2k} \frac{dp}{dx} + \mu_0 \left(\frac{\partial u}{\partial y}\right)_0 = 0. \tag{2.24}$$

Putting (1.8) in dimensionless form, we have*

$$G_1 \frac{dw}{dx} + \frac{k - 1}{2k} \frac{dp}{dx} + \frac{\zeta}{8} G_1 w\widetilde{R}e = 0. \tag{2.25}$$

(2.24) and (2.25) are the same if we assume that $w = u_m$ and put

$$\zeta = \frac{8\mu_0}{\widetilde{R}eG_1 u_m} \left(\frac{\partial u}{\partial y}\right)_0. \tag{2.26}$$

This expression for ζ will be used in future.

4. Relations at the Axis of the Channel

The method given below is based on (2.16), (2.19), and (2.20), together with (2.8) and (2.9) as written for the axis.

The latter equations are obtained by putting $u = u_1$ and $v = v_1$ in (2.8) and (2.9); $v = 0$, on account of the symmetry, so

---

*Here the hydraulic diameter is $D = 4h$, which must not be forgotten.

$$\rho_1 u_1 \frac{du_1}{dx} + \frac{k - 1}{2k}\frac{dp}{dx} = \left\{ \frac{\partial}{\partial y}\left[(\mu + \epsilon)\frac{\partial u}{\partial y} \right] \right\}_1,$$

$$\rho_1 u_1 \frac{d\Theta_1}{dx} = \left\{ \frac{\partial}{\partial y}\left[(\mu + \epsilon)\frac{\partial \Theta}{\partial y} \right] \right\}_1.$$

Prandtl's theory [24] gives ϵ as

$$\epsilon = \rho l^2 \frac{\partial u}{\partial y},$$

in which l is the length of the displacement path.

But $(\partial u/\partial y)_1 = 0$, so $\epsilon_1 = 0$, and*

$$\left\{ \frac{\partial}{\partial y}\left[(\mu + \epsilon)\frac{\partial u}{\partial y} \right] \right\}_1 = \left(\frac{\partial \mu}{\partial y} + \frac{\partial \epsilon}{\partial y} \right)_1 \left(\frac{\partial u}{\partial y} \right)_1 +$$

$$+ (\mu_1 + \epsilon_1)\left(\frac{\partial^2 u}{\partial y^2} \right)_1 = \mu_1 \left(\frac{\partial^2 u}{\partial y^2} \right)_1.$$

But $(\partial\Theta/\partial y)_1 = 0$, on account of the symmetry, so we have

---

*All deductions still apply if $\epsilon_1 \neq 0$.

$$\left\{ \frac{\partial}{\partial y} \left[(\mu + \epsilon) \frac{\partial \Theta}{\partial y} \right] \right\}_1 = \mu_1 \left(\frac{\partial^2 \Theta}{\partial y^2} \right)_1$$

and the equations for the axis can be put as

$$\rho_1 u_1 \frac{du_1}{dx} + \frac{k - 1}{2k} \frac{dp}{dx} = \mu_1 \left(\frac{\partial^2 u}{\partial y^2} \right)_1, \tag{2.27}$$

$$\rho_1 u_1 \frac{d\Theta_1}{dx} = \mu_1 \left(\frac{\partial^2 \Theta}{\partial y^2} \right)_1. \tag{2.28}$$

The derivation of the right-hand sides of these equations will be dealt with in the presentation of the computation method.

5. Relations near the Wall

Simple power functions will be used for the velocity and temperature profiles, in order to provide an approximate solution:

$$u = u_1 y^n, \tag{2.29}$$

$$\Theta - \Theta_0 = (\Theta_1 - \Theta_0)\, y^m, \tag{2.30}$$

in which n and m are form parameters (functions of x), with $0 \leq n \leq 1$ and $0 \leq m \leq 1$.

The profile of (2.29) cannot be extended right up to the wall, since $\partial u/\partial y = u_1 n y^{n-1} \rightarrow \infty$ for $y \rightarrow 0$.

Following Prandtl, we assume that $\epsilon = 0$ near the wall in the viscous sublayer; here inertial and pressure forces are small relative to the viscous ones. Then (2.1) gives us approximately that

$$\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0,$$

so, with μ = constant, we have a linear velocity distribution near the wall:

$$u = u_\delta \frac{y}{\delta},$$

in which δ is the thickness of the viscous sublayer.

We cannot put μ = constant for the sublayer if heat transfer is important, for μ is a function of temperature, and this may vary substantially near the wall. In this case

$$\mu \frac{\partial u}{\partial y} = \mu_0 \left(\frac{\partial u}{\partial y}\right)_0,$$

i.e., the frictional stress is constant within the sublayer.

The last equation gives

$$u = \frac{\mu_0}{\bar{\mu}_y}\left(\frac{\partial u}{\partial y}\right)_0 y, \qquad \text{(a)}$$

in the symbols

$$\frac{1}{\bar{\mu}_y} = \frac{1}{y}\int_0^y \frac{dy}{\mu} . \qquad \text{(b)}$$

We put $y = \delta$ in (a) to get

$$u_\delta = \frac{\mu_0}{\bar{\mu}_\delta}\left(\frac{\partial u}{\partial y}\right)_0 \delta,$$

whence the frictional stress at the wall is found as

$$\mu_0\left(\frac{\partial u}{\partial y}\right)_0 = \frac{u_\delta}{\delta}\bar{\mu}_\delta. \tag{2.31}$$

Here $\bar{\mu}_\delta$ is a mean value for the viscosity of the sublayer.

From (a) and (b) we have

$$u = \frac{\bar{\mu}_\delta}{\bar{\mu}_y} u_\delta \frac{y}{\delta}.$$

The velocity profile in the sublayer is not linear in this case.

The thickness of the sublayer is found from the following argument. The Prandtl-Karman theory gives for M small and no heat transfer that δ is defined by*

$$\frac{u_\delta \delta}{\nu} = Re_\delta = \text{const} \tag{2.32}$$

(u_δ, δ, and ν are dimensional here). We have in dimensionless form

*This constant is denoted by α^2 in turbulence theory, but α is used for the heat-transfer factor here.

that

$$\tilde{Re}\,\frac{u_\delta \delta}{\nu} = Re_\delta . \qquad (c)$$

We assume that for M large and for heat transfer present the Re_δ is the same if as our ν in (c) we put $\bar{\nu} = \bar{\mu}_\delta/\bar{\rho}$, in which $\bar{\mu}_\delta$ is the mean viscosity of the sublayer (see above), while $\bar{\rho}$ is the density at a temperature $\bar{T}_\delta$ corresponding to $\bar{\mu}_\delta$.

Then δ is given by

$$\tilde{Re}\,\frac{u_\delta \delta \bar{\rho}}{\bar{\mu}_\delta} = Re_\delta . \qquad (d)$$

But $\bar{\rho} = p/T$, so, putting the relation of viscosity to temperature as

$$\mu = T^\beta \qquad (2.33)$$

and using (2.29), we have

$$\tilde{Re}\,\frac{u_1 \delta^{n+1} p}{\bar{\mu}_\delta^{1+\frac{1}{\beta}}} = Re_\delta ,$$

and so

$$\delta = \left(\frac{Re_\delta \bar{\mu}_\delta^{1+\frac{1}{\beta}}}{\tilde{Re}\, u_1 p}\right)^{\frac{1}{n+1}} \qquad (2.34)$$

Section 8 of this chapter deals with the determination of Re_δ.

We need to know the distribution of μ in the sublayer in order to find δ and the velocity profile there. We assume that $1/\mu$ is a linear function of y within the sublayer:

$$\frac{1}{\mu} = \frac{1}{\mu_0} + \left(\frac{1}{\mu_\delta} - \frac{1}{\mu_0}\right)\frac{y}{\delta},$$

so

$$\frac{1}{\bar{\mu}_\delta} = \frac{1}{\delta}\int_0^\delta \frac{dy}{\mu} = \frac{1}{\mu_0} + \frac{1}{2}\left(\frac{1}{\mu_\delta} - \frac{1}{\mu_0}\right). \tag{2.35}$$

An analogous sublayer occurs for the stagnation temperature; with Pr = 1 we may [15,40] assume that the thickness of this is also δ, so at the wall we have, by analogy with (2.31), that

$$\mu_0\left(\frac{\partial\Theta}{\partial y}\right)_0 = \frac{\Theta_\delta - \Theta_0}{\delta}\bar{\mu}_\delta. \tag{2.36}$$

Successive approximation may be used to find δ from (2.32) and $\bar{\mu}_\delta$ from (2.35).

We first put $\bar{\mu}_\delta = \mu_0$ in (2.35) to get from (2.33) that $\bar{\mu}_\delta = \Theta_0^\beta$; from this $\bar{\mu}_\delta$ we find from (2.34) the first approximation $\delta^{(1)}$, so we have

$$T_\delta(1) = \Theta_\delta(1) - u^2_\delta(1).$$

From (2.29) and (2.30) we have

$$T_{\delta(1)} = \Theta_0 + (\Theta_1 - \Theta_0)\delta^{(1)^m} - u_1^2\delta^{(1)^{2n}}.$$

From (2.33)

$$\mu_{\delta(1)} = T^{\beta}_{\delta(1)} .$$

From this $\mu_{\delta(1)}$ we can find a second approximation $\delta^{(2)}$. The process is continued until the differences between successive approximations become sufficiently small.

Note. The calculation of δ and $\bar{\mu}_\delta$ can be improved if we use for $1/\mu$ the approximating function

$$\frac{1}{\mu} = \frac{1}{\mu_0} + \left(\frac{1}{\mu_\delta} - \frac{1}{\mu_0}\right)\left(\frac{y}{\delta}\right)^{\omega},$$

in which ω is a constant that can be found from the following argument. We find δ by the above method and require that for $y/\delta = 1/2$ we have

$$T^{\beta} = \mu.$$

This gives us an equation in the unknown ω, which is found and is used to revise δ and $\bar{\mu}_\delta$, which are used to revise ω, and so on.

6. Expressions for the Coefficients of Resistance and Heat Transfer in Terms of the Boundary-Layer Parameters

We now need expressions for ζ and α; the first (the resistance coefficient) has been found above as (2.26). From (2.29) and (2.31) we have

$$\zeta = \frac{8\bar{\mu}_\delta}{\widetilde{\mathrm{Re}}G_1} \frac{u_1}{u_m} \delta^{n-1}. \tag{2.37}$$

Now α is, by definition, the ratio of the heat flux at the wall to the difference between the wall temperature and the mean stagnation temperature [all quantities in the equations up to (2.38) are dimensional, apart from the numbers]:

$$\alpha = \frac{q_c}{\Theta_0 - \Theta_m} .$$

We now introduce the dimensionless heat-transfer factor in the form of Stanton's number:

$$\mathrm{ST} = \frac{F_1}{G_1} \frac{q_0}{c_p(\Theta_0 - \Theta_m)} .$$

But $\mathrm{Pr} = 1$, so $c_p = \lambda_0/\mu_0$, and the last equation is replaced by the following by virtue of (2.15):

$$\mathrm{ST} = -\frac{F_1}{G_1} \frac{\mu_0}{\Theta_0 - \Theta_m} \left(\frac{\partial\Theta}{\partial y}\right)_0 ,$$

or, after converting the right-hand side to dimensionless form:

$$ST = -\frac{1}{\tilde{R}eG_1}\frac{\mu_0}{\Theta_0 - \Theta_m}\left(\frac{\partial\Theta}{\partial y}\right)_0. \tag{2.38}$$

That is, we must know $(\partial u/\partial y)_0$, $(\partial\Theta/\partial y)_0$, u_m, and Θ_m in order to determine ζ and St.

Expressions for the two partial derivatives in terms of m, n, u_1, Θ, Θ_0, and δ have been derived in the previous section.

The expressions are simplified by the use of the following symbols:

$$I_1 = \int_0^1 \frac{u}{\Theta - u^2}\,dy, \tag{2.39}$$

$$I_2 = \int_0^1 \frac{u^2}{\Theta - u^2}\,dy, \tag{2.40}$$

$$I_3 = \int_0^1 \frac{u\Theta}{\Theta - u^2}\,dy. \tag{2.41}$$

(2.11) and (2.12) enable us to put (2.17), (2.22), and (2.21) as

$$G_1 = \int_0^1 \rho u dy = p\int_0^1 \frac{u}{\Theta - u^2}\,dy = pI_1, \tag{2.42}$$

$$G_1 u_m = \int_0^1 \rho u^2 dy = p \int_0^1 \frac{u^2}{\Theta - u^2} dy = pI_2, \tag{2.43}$$

$$G_1 \Theta_m = \int_0^1 \rho u\Theta dy = p \int_0^1 \frac{u\Theta}{\Theta - u^2} dy = pI_3. \tag{2.44}$$

(2.42) and (2.43) give

$$u_m = \frac{I_2}{I_1}. \tag{2.45}$$

(2.42) and (2.44) give

$$\Theta_0 - \Theta_m = \frac{I_1\Theta_0 - I_3}{I_1}.$$

The mean speed of (2.45) is substituted into (2.37) to give

$$\zeta = \frac{8\bar{\mu}_\delta}{\tilde{R}eG_1} \frac{I_1}{I_2} u_1 \delta^{n-1} . \tag{2.46}$$

Similarly, (2.38), (2.36), and the expression for $\Theta_0 - \Theta_m$ give

$$ST = \frac{\bar{\mu}_\delta}{\tilde{R}eG_1} \frac{I_1}{I_1\Theta_0 - I_3} (\Theta_0 - \Theta_1)\delta^{m-1} \tag{2.47}$$

That is, we must be able to compute I_1, I_2, and I_3 as well as know u_1, Θ_1, δ, m, and n in order to determine ζ and St; that is, we must find the pressure, mean velocity, mean stagnation temperature, and G_1 from (2.42)-(2.44), in accordance with given initial conditions.

The following sections deal with these.

7. Calculation of Flow Rate, Pressure, and Mean Gas Parameters

First we find I_1, I_2, and I_3 as given by (2.39)-(2.41). To simplify the treatment we assume that (2.29) and (2.30) apply right out to the wall; this cannot introduce any substantial error, on account of the small thickness of the viscous sublayer*.

Now $u^2 < \Theta = u^2 + T$, so $1/(\Theta - u^2)$ can be put in the form of a convergent series:

$$\frac{1}{\Theta - u^2} = \Theta^{-1} (1 + u^2\Theta^{-1} + u^4\Theta^{-2} + \ldots).$$

We put

$$\int_0^1 u^i\Theta^j dy = I^{(i,j)},$$

to get

$$I_1 = I^{(1,-1)} + I^{(3,-2)} + I^{(5,-3)} + \ldots, \qquad (2.48)$$

*See Appendix II.

$$I_2 = I^{(2,-1)} + I^{(4,-2)} + I^{(6,-3)} + \dots, \tag{2.49}$$

$$I_3 = I^{(1,0)} + I^{(3,-1)} + I^{(5,-2)} + \dots . \tag{2.50}$$

It remains to find the integrals $I^{(i,j)}$. We have from (2.29) and (2.30) that

$$u^i\Theta^j = u_1^i y^{ni}[\Theta_0 + (\Theta_1 - \Theta_0)\, y^m]^j = u_1^i y^{ni}\Theta_0^j \left[\, 1 + \left(\frac{\Theta_1}{\Theta_0} - 1\right) y^m \,\right]^j .$$

We consider only the case of heat inflow; $0 < \Theta_1/\Theta_0 \leqslant 1$, so $|1 - \Theta_1/\Theta_0| < 1$, and we have the convergent series

$$\left[\, 1 + \left(\frac{\Theta_1}{\Theta_0} - 1\right) y^m \,\right]^j = 1 + \sum_{k-1}^{\infty} \binom{j}{k} \left(\frac{\Theta_1}{\Theta_0} - 1\right)^k y^{mk},$$

in which

$$\binom{j}{k} = \frac{j\,(j-1)\,(j-2)\,\dots\,(j-k+1)}{k!},$$

so*

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*See Appendix I regarding the operations on the series.

$$I^{(i,j)} = u_1^i \Theta_0^j \left[\frac{1}{ni + 1} + \sum_{k=1}^{\infty} \frac{\binom{j}{k}\left(\frac{\Theta_1}{\Theta_0} - 1\right)^k}{ni + mk + 1} \right] \tag{2.51}$$

I_1, I_2, and I_3 are then known.

We must be given u_{1i}, Θ_{1i}, n_i, and m_i (for the initial channel cross-section); from these we have from (2.42) the gas flow rate:

$$G_1 = p_i I_{1i}, \tag{2.52}$$

in which

$$p_i = \Pi\left(u_{1i} \sqrt{\frac{k + 1}{k - 1}} \right).$$

From the flow rate we have the pressure in any channel cross-section from (2.42) as

$$p = \frac{G_1}{I_1}. \tag{2.53}$$

(2.43) and (2.44) are then used to find the mean speed and mean stagnation temperature of the gas. This method of finding the mean parameters enables us (in accordance with the statement at the end of section 3) to ensure that the quantity of motion, enthalpy, and flow rate are the same in the actual and assumed one-dimensional flows. In addition, the pressure, and hence the momentum, are the same. It has been shown [35] that this method of finding the mean is one of the two best.

The mean density and temperature must be found directly from (1.2) and (1.14):

$$\rho_m = \frac{G_1}{u_m}, \tag{2.54}$$

$$T_m = \Theta_m - u_m^2 . \tag{2.55}$$

8. Calculation of the Thickness of the Viscous Sublayer

This thickness δ is expressed in section 5 in terms of Re_δ in (2.34). We determine Re_δ on the following basis. We require that $u_1 \ll [(k - 1)/(k + 1)]^{1/2}$; that is, that (2.37) should coincide with Blasius's formula

$$\zeta = 0.3164\ Re^{-\frac{1}{4}} \tag{2.56}$$

for speeds well below sonic in the absence of heat transfer ($m = 0$, $\Theta_0 = \Theta_1 = 1$), this being applicable for $Re < 10^5$. In (2.37) we must put $n = 1/7$, which corresponds to a Blasius velocity profile.

Re must be determined for the mean parameters by reference to the hydraulic diameter, which is twice the width of the channel; the linear dimension is half the width in the calculation of Re, so

$$\widetilde{Re} = \frac{1}{4} Re \frac{\mu_m}{u_m \rho_m}. \tag{2.57}$$

(2.33), (2.54), and (2.55) give

$$\tilde{Re} = \frac{1}{4}\frac{Re}{G_1}(\Theta_m - u_m^2)^\beta .$$

(2.34), taken with the above and with (2.53), gives

$$\delta = \left[\frac{4Re_\delta}{Re} \bar{\mu}_\delta^{\frac{1}{\beta}+1} \frac{I_1}{u_1} (\Theta_m - u_m^2)^\beta \right]^{\frac{1}{n+1}}$$

These values of Re and δ are inserted in (2.37) to give after manipulation that

$$\zeta = 32 \frac{u_1}{u_m} (\Theta_m - u_m^2)^{\beta \frac{2n}{n+1}} \left(4Re_\delta \frac{I_1}{u_1} \right)^{\frac{n-1}{n+1}} \bar{\mu}_\delta^{\frac{(2\beta+1)n-1}{\beta(n+1)}} Re^{-\frac{2n}{n+1}} . \quad (2.58)$$

With $u_1 \ll [(k-1)/(k+1)]^{1/2}$, we have for $n = 1/7$ from (2.48), (2.49), and (2.51) that

$$I_1 = \frac{u_1}{n+1} = \frac{7}{8} u_1, \quad I_2 = \frac{u_1^2}{2n+1} = \frac{7}{9} u_1^2 .$$

From (2.45)

$$u_m = \frac{I_2}{I_1} = \frac{n+1}{2n+1} u_1 = \frac{8}{9} u_1 .$$

We can put $\bar{\mu}_\delta = 1$ for low speeds in adiabatic flows. In addition,

$$\Theta_m - u_m^2 = 1 - u_m^2 \approx 1.$$

Then for low speeds and adiabatic flows we can put (2.58) for $Re < 10^5$ in the form

$$\zeta = 36\left(\frac{7}{2}Re_\delta\right)^{-\frac{3}{4}} Re^{-\frac{1}{4}}.$$

Comparison with the Blasius formula (2.56) shows that

$$36\left(\frac{7}{2}Re_\delta\right)^{-\frac{3}{4}} = 0.3164,$$

whence

$$Re_\delta = \frac{2}{7}\left(\frac{36}{0.3164}\right)^{\frac{4}{3}} = 157^*. \qquad (2.59)$$

This gives us Re_δ and δ, so ζ can be found from (2.37), St from (2.48), and also the other flow parameters, if m, n, u_1, and Θ_1 are known.

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*Nikuradze's experiments [52] give $Re_\delta = 132$, but I have retained $Re_\delta = 157$ here in order to be able to use Blasius's formula.

Before we consider how to compute these we must consider the conditions under which Reynolds analogy is applicable. This enables us to choose m and n correctly for the initial section and also to obtain additional data needed in the development of the method.

9. Reynolds Analogy

We determine the ratio of ζ to St; dividing (2.46) by (2.47), we have

$$\frac{\zeta}{St} = 8 \frac{u_1}{\Theta_0 - \Theta_1} \frac{\Theta_0 I_1 - I_3}{I_2} \delta^{n-m}. \tag{2.60}$$

(2.29), (2.30), and (2.39)-(2.41) give

$$\Theta_0 I_1 - I_3 = \Theta_0 \int_0^1 \frac{u}{\Theta - u^2} dy - \int_0^1 u \frac{\Theta_0 + (\Theta_1 - \Theta_0) y^m}{\Theta - u^2} dy.$$

or

$$\Theta_0 I_1 - I_3 = (\Theta_0 - \Theta_1) u_1 \int_0^1 \frac{y^{m+n}}{\Theta - u^2} dy. \tag{2.61}$$

In addition

$$I_2 = \int_0^1 \frac{u^2}{\Theta - u^2} dy = u_1^2 \int_0^1 \frac{y^{2n}}{\Theta - u^2} dy. \tag{2.62}$$

We assume that the velocity and temperature profiles are similar (i.e., that $m = n$). Substituting (2.61) and (2.62) into (2.60) and putting $m = n$, we have

$$\frac{\zeta}{St} = 8. \tag{2.63}$$

This direct proportionality is called the basic relationship of Reynolds analogy or the basic relation of the hydrodynamic theory of heat transfer [7]. It was first derived in 1874 by Reynolds on the assumption that heat and momentum are transferred in identical fashion [55].

The conditions for the two profiles to be similar are readily found from (2.8) and (2.9) (boundary-layer equations), which imply directly that for $Pr = 1$ (the case considered here) we must have $dp/dx = 0$ and Θ_0 = constant if the two are to be similar.

This means that the analogy applies, strictly speaking, only when the gas flows around an isothermal plate; but experiment shows that it applies quite closely for gases and liquids flowing in pipes [36,37,42].

The effects of temperature variation in the plate on the relation between ζ and St may be estimated [30] from the results of [10]. It is shown that for $dp/dx = 0$, $Pr = 1$, and $\mu = T$ equations (2.8)-(2.12) for the case of a laminar boundary layer ($\varepsilon \equiv 0$) become ordinary differential equations incorporating the change in temperature of the wall:

$$\varphi''' + 2\varphi\varphi'' = 0, \tag{2.64}$$

$$g'' + 2\varphi g' + 4\omega\varphi' g = 0, \tag{2.65}$$

with boundary conditions

$$\varphi(0) = \varphi'(0) = 0;\ \varphi'(\infty) = 1;\ g(0) = 1,\ g(\infty) = 0. \tag{2.66}$$

Here φ and g are functions of the dimensionless variable ξ; u and Θ

are proportional respectively to $U\varphi'$ and $T(\Theta_0 - \Theta_\infty)g$, in which $U = u/_{y=\infty}$ (the proportionality factor is the same for both).

The ω appearing in (2.65) is the (constant) power appearing in

$$\frac{\Theta_0 - \Theta_\infty}{\Theta_\infty} = A\bar{x}^{\omega}, \tag{2.67}$$

for the relation of Θ_0 (plate temperature) to $\bar{x} = x/L$ (x and L are dimensional, L being the characteristic length).

(2.64) subject to (2.66) is Blasius's equation [43]; the solution is tabulated, which facilitates computation.

We use for ζ and St expressions suitable for the exterior problem:

$$\zeta = \frac{8\mu_0}{\rho_\infty U^2}\left(\frac{\partial u}{\partial y}\right)_0, \quad St = \frac{\mu_0}{\rho_\infty U}\frac{\left(\frac{\partial \Theta}{\partial y}\right)_0}{\Theta_\infty - \Theta_0},$$

and it is then simple to find

$$\frac{\zeta}{ST} = 8\,\frac{\varphi''(0)}{g'(0)}. \tag{2.68}$$

We integrate (2.65) as an ordinary linear differential equation in $g'(\xi)$ to get

$$g'(\xi) = \exp\left(- \int_0^\xi 2\varphi d\xi \right)\left[g'(0) - 4\omega \int_0^\xi \varphi' g \exp\left(\int_0^\xi 2\varphi d\xi \right) d\xi\right] \quad (2.69)$$

Now (2.64) gives $2\varphi = - \varphi'''/\varphi''$, so

$$\int_0^\xi 2\varphi d\xi = - \int_0^\xi \frac{\varphi'''}{\varphi''} d\xi = - \ln \frac{\varphi''(\xi)}{\varphi''(0)} ,$$

and from (2.69) we have

$$g'(\xi) = \frac{\varphi''(\xi)}{\varphi''(0)} \left[g'(0) - 4\omega\varphi''(0) \int_0^\xi \frac{\varphi' g}{\varphi''} d\xi \right] .$$

We integrate this and use the fact that $g(0) = 1$ by virtue of the boundary condition of (2.66) to get

$$g(\xi) = \frac{\varphi'(\xi)}{\varphi''(0)} g'(0) \qquad \omega \int_0^\xi \varphi''(\xi) F(\xi) d\xi + 1, \qquad (2.70)$$

in which the symbol is

$$F(\xi) = \int_0^\xi \frac{\varphi' g}{\varphi''} d\xi.$$

We put $\xi = \infty$ in (2.70) and use the fact that $g(\infty) = 0$ and $\varphi'(\infty) = 1$ to get

$$g(\infty) = \frac{g'(0)}{\varphi''(0)} - 4\omega \int_0^\infty \varphi''(\xi)\, F(\xi)\, d\xi + 1 = 0,$$

whence

$$\frac{g'(0)}{\varphi''(0)} = 4\omega \int_0^\infty \varphi''(\xi)\, F(\xi)\, d\xi - 1. \tag{2.71}$$

In place of (2.70) we now have

$$g(\xi) = 1 - \varphi'(\xi) - 4\omega \left[\int_0^\xi \varphi''(\xi)\, F(\xi)\, d\xi - \varphi'(\xi) \int_0^\infty \varphi''(\xi)\, F(\xi)\, d\xi \right]. \tag{2.72}$$

Successive approximation may be used to find $g(\xi)$ if ω is reasonably small. As a first approximation we take $g = g_I(\xi) = 1 - \varphi'(\xi)$, which is inserted on the right in (2.72) to give $g_{II}(\xi)$ as second approximation, and so on. From $g(\xi)$ we find $g'(0)/\varphi''(0)$ from (2.71). Taking only the second approximation for $g(\xi)$, we have

$$-\frac{g'(0)}{\varphi''(0)} \approx 1 - 1.06\omega + 0.0448\omega^2,$$

and so from (2.68)

$$\frac{\zeta}{St} \approx \frac{8}{1 - 1.06\omega + 0.0488\omega^2} . \qquad (2.73)$$

Then the analogy applies for $|\omega|$ sufficiently small; ξ/St for any given ω is constant over the plate: $\zeta/ST = C =$ constant. But even quite small changes in ω cause C to vary substantially (Fig. 11).

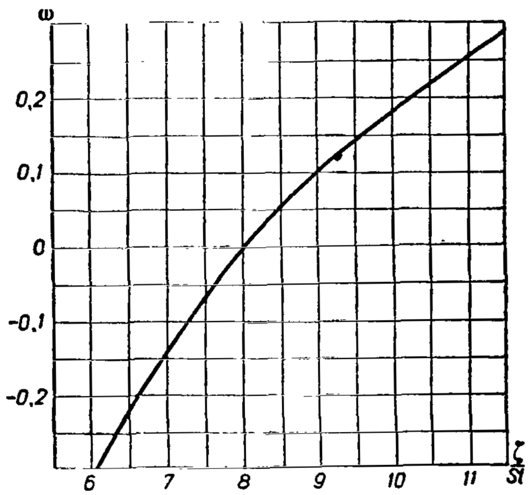


Fig. 11.

Relation of the constant $\zeta/St =$ $= C$ to ω.

This means that there can be substantial deviations from the analogy if the temperature distribution in the plate does not follow a power law, so careful allowance must be made for temperature variation in the walls in developing the method.

10. Approximate Method of Computation for a Given Temperature Distribution at the Wall

The initial equations to be used are (2.16), (2.19), (2.20), (2.27), and (2.28). Use of (2.11), (2.12), and (2.42)-(2.44) gives us a system of five equations:

$$\frac{d}{dx}(pI_1) = 0,$$

$$\frac{d}{dx}(pI_2) + \frac{k-1}{2k}\frac{dp}{dx} = -\mu_0\left(\frac{\partial u}{\partial y}\right)_0,$$

$$\frac{d}{dx}(pI_3) = -\mu_0\left(\frac{\partial \Theta}{\partial y}\right)_0, \tag{2.74}$$

$$\frac{pu_1}{\Theta_1 - u_1^2}\frac{du_1}{dx} + \frac{k-1}{2k}\frac{dp}{dx} = \mu_1\left(\frac{\partial^2 u}{\partial y^2}\right)_1,$$

$$\frac{pu_1}{\Theta_1 - u_1^2}\frac{d\Theta_1}{dx} = \mu_1\left(\frac{\partial^2 \Theta}{\partial y^2}\right)_1$$

in the five unknowns p, u_1, Θ_1, m, and n, whose values must be given at the channel inlet (i.e., at $x = 0$).

We eliminate p from the system; the first equation in (2.74) gives

$$pI_1 = G_1 = \text{const},$$

or

$$p = \frac{G_1}{I_1} . \tag{2.53}$$

We differentiate the products in the first three equations of (2.74) and substitute from (2.53) for p to get

$$\frac{dI_1}{dx} + \frac{I_1^2}{G_1}\frac{dp}{dx} = 0 ,$$

$$\frac{dI_2}{dx} + \frac{I_1\left(I_2 + \frac{k-1}{2k}\right)}{G_1}\frac{dp}{dx} = -\frac{I_1}{G_1}\mu_0\left(\frac{\partial u}{\partial y}\right)_0, \tag{2.75}$$

$$\frac{dI_3}{dx} + \frac{I_1 I_3}{G_1}\frac{dp}{dx} = -\frac{I_1}{G_1}\mu_0\left(\frac{\partial \Theta}{\partial y}\right)_0 .$$

The fifth equation in (2.74) becomes as follows when (2.53) is used:

$$\frac{d\Theta_1}{dx} = \frac{I_1}{G_1}\frac{\Theta_1 - u_1^2}{u_1}\mu_1\left(\frac{\partial^2 \Theta}{\partial y^2}\right)_1 . \tag{2.76}$$

The fourth equation of (2.74), together with (2.53), gives

$$\frac{dp}{dx} = -\frac{2k}{k-1}\frac{G_1}{I_1}\frac{u_1}{\Theta_1 - u_1^2}\frac{du_1}{dx} + \frac{2k}{k-1}\mu_1\left(\frac{\partial^2 u}{\partial y^2}\right)_1. \qquad (2.77)$$

This value of dp/dx is inserted in (2.75); the result is combined with (2.76) to give us a system of four equations in the unknowns u_1, n, Θ_1, and m:

$$\frac{dI_1}{dx} - \frac{2k}{k-1} I_1 \frac{u_1}{\Theta_1 - u_1^2}\frac{du_1}{dx} = -\frac{2k}{k-1}\frac{I_1^2}{G_1}\mu_1\left(\frac{\partial^2 u}{\partial y^2}\right)_1,$$

$$\frac{dI_2}{dx} - \left(\frac{2k}{k-1} I_2 + 1\right)\frac{u_1}{\Theta_1 - u_1^2}\frac{du_1}{dx} = -\frac{I_1}{G_1}\mu_0\left(\frac{\partial u}{\partial y}\right)_0 -$$

$$-\frac{I_1\left(\frac{2k}{k-1} I_2 + 1\right)}{G_1}\mu_1\left(\frac{\partial^2 u}{\partial y^2}\right)_1, \qquad (2.78)$$

$$\frac{dI_3}{dx} - \frac{2k}{k-1} I_3 \frac{u_1}{\Theta_1 - u_1^2}\frac{du_1}{dx} = -\frac{I_1}{G_1}\mu_0\left(\frac{\partial \Theta}{\partial y}\right)_0 - \frac{2k}{k-1}\frac{I_1 I_3}{G_1}\mu_1\left(\frac{\partial^2 u}{\partial y^2}\right)_1,$$

$$\frac{d\Theta_1}{dx} = \frac{I_1}{G_1}\frac{\Theta_1 - u_1^2}{u_1}\mu_1\left(\frac{\partial^2 \Theta}{\partial y^2}\right)_1.$$

For convenience in further operations we change the variables by putting

$u_1 \equiv z_1$, $n \equiv z_2$, $\Theta_1 \equiv z_3$, and $m \equiv z_4$; I_1, I_2, and I_3 are functions of z_1, z_2, z_3, and z_4, and also of Θ_0, which in general is a given function of x, so

$$\frac{dI_k}{d_x} = \sum_{j=1}^{4} \frac{\partial I_k}{\partial z_j} \frac{dz_j}{dx} + \frac{\partial I_k}{\partial \Theta_0} \frac{d\Theta_0}{dx}, \; k = 1, 2, 3.$$

Then (2.78) may be put as

$$N_{i_1} \frac{dz_1}{dx} + N_{i_2} \frac{dz_2}{dx} + N_{i_3} \frac{dz_3}{dx} + N_{i4} \frac{dz_4}{dx} = N_i. \tag{2.79}$$

$$(i = 1, 2, 3, 4).$$

(2.79) is quasilinear in the derivatives of the z with respect to x. An analogous system of equations has been derived [2] for flow in nozzles in the presence of a flow core.

The coefficients N_{ik} of (2.79) are

$$N_{11} = \frac{\partial I_1}{\partial z_1} - \frac{2k}{k-1} \frac{I_1}{z_3 - z_1^2} z_1,$$

$$N_{21} = \frac{\partial I_2}{\partial z_1} - \frac{\frac{2k}{k-1} I_2 + 1}{z_3 - z_1^2} z_1,$$

$$N_{31} = \frac{\partial I_3}{\partial z_1} - \frac{2k}{k-1} \frac{I_3}{z_3 - z_1^2} z_1 , \tag{2.80}$$

$$N_{ij} = \frac{\partial I_i}{\partial z_j}, \ (i = 1, 2, 3; \ j = 2, 3, 4),$$

$$N_{43} = 1; \ N_{41} = N_{42} = N_{44} = 0.$$

The independent terms in (2.79) are

$$N_1 = - \frac{2k}{k-1} \frac{I_1^2}{G_1} \mu_1 \left(\frac{\partial^2 u}{\partial y^2}\right)_1 - \frac{\partial I_1}{\partial \Theta_0} \frac{d\Theta_0}{dx} ,$$

$$N_2 = - \frac{I_1}{G_1} \mu_0 \left(\frac{\partial u}{\partial y}\right)_0 - I_1 \frac{\frac{2k}{k-1} I_2 + 1}{G_1} \mu_1 \left(\frac{\partial^2 u}{\partial y^2}\right)_1 - \frac{\partial I_2}{\partial \Theta_0} \frac{d\Theta_0}{dx}, \tag{2.81}$$

$$N_3 = - \frac{I_1}{G_1} \mu_0 \left(\frac{\partial \Theta}{\partial y}\right)_0 - \frac{2k}{k-1} \frac{I_1 I_3}{G_1} \mu_1 \left(\frac{\partial^2 u}{\partial y^2}\right)_1 - \frac{\partial I_3}{\partial \Theta_0} \frac{d\Theta_0}{dx} ,$$

$$N_4 = \frac{I_1}{G_1} \frac{z_3 - z_1^2}{z_1} \mu_1 \left(\frac{\partial^2 \Theta}{\partial y^2}\right)_1 .$$

The N_{ik} are determined in terms of the I_i and the $\partial I_i/\partial z_k$, in accordance with (2.80); I_1, I_2, and I_3 have been found [formulas (2.48)-(2.50) of section 7].

We need to know the $I^{i,j}$ and their derivatives in order to find the I_i and their derivatives. Formula (2.51) gives the $I^{i,j}$, and we now rewrite this expression in the new symbols as

$$I^{(i,j)} = z_1^i \Theta_0^j \left[\frac{1}{z_2 i + 1} + \sum_{k=1}^{\infty} \frac{\binom{j}{k}\left(\frac{z_3}{\Theta_0} - 1\right)^k}{iz_2 + kz_4 + 1} \right]$$

This is differentiated with respect to z_i to give

$$\frac{\partial I^{(i,j)}}{\partial z_1} = \frac{i}{z_1} I^{(i,j)};$$

$$\frac{\partial I^{(i,j)}}{\partial z_2} = - Z_1^i \Theta_0^j i \left[\frac{1}{(z_2 i + 1)^2} + \sum_{k=1} \frac{\binom{j}{k}\left(\frac{z_3}{\Theta_0} - 1\right)^k}{(iz_2 + kz_4 + 1)^2} \right];$$

$$\frac{\partial I^{(i,j)}}{\partial z_3} = z_1^i \Theta_0^{j-1} \sum_{k=1}^{\infty} k \binom{j}{k} \frac{\left(\frac{z_3}{\Theta_0} - 1\right)^{k-1}}{iz_2 + kz_4 + 1};$$

$$\frac{\partial I^{(i,j)}}{\partial z_4} = - z_1^i \Theta_0^j \sum_{k=1}^{\infty} k \binom{j}{k} \frac{\left(\frac{z_3}{\Theta_0} - 1\right)^k}{(iz_2 + kz_4 + 1)^2} .$$

Differentiation with respect to Θ_0 gives

$$\frac{\partial I^{(i,j)}}{\partial \Theta_0} = \frac{j}{\Theta_0} I^{(i,j)} - z_1^i \Theta_0^{j-2} z_3 \sum_{k=1}^{\infty} k \binom{j}{k} \frac{\left(\frac{z_3}{\Theta_0} - 1\right)^{k-1}}{iz_3 + kz_4 + 1} .$$

The $\partial I^{(i,j)}/\partial \Theta_0$ enable us to use (2.48)-(2.50) to find the $\partial I_k/\partial \Theta_0$ that appear in (2.81). The $(\partial u/\partial y)_0$ and $(\partial \Theta/\partial y)_0$ appearing there are given by (2.31) and (2.36), so there remain only $(\partial^2 u/\partial y^2)_1$ and $(\partial^2 \Theta/\partial y^2)_1$, which require special discussion*.

We put

$$\mu_1 \left(\frac{\partial^2 u}{\partial y^2}\right)_1 = - f\mu_0 \left(\frac{\partial u}{\partial y}\right)_0 . \tag{2.82}$$

*These cannot be found from (2.29) and (2.30); although power profiles approach the actual ones closely, they cannot be continued right up to the axis (or up to the wall), because they have cusps at the axis, whereas the real ones do not.

(2.27) becomes

$$\rho_1 u_1 \frac{du_1}{dx} + \frac{k-1}{2k}\frac{dp}{dx} + f\mu_0 \left(\frac{\partial u}{\partial y}\right)_0 = 0.$$

Comparison with (2.19) shows that f for $[\partial(\rho v)/\partial y]_1 = 0$ characterizes the change along the length in the difference between the mean quantity of motion and that at the axis. Continuity implies that $-d(\rho_1 u_1)/dx = [\partial(\rho v)/\partial y]_1$, so

$$\rho_1 u_1 \frac{du_1}{dx} = \frac{d}{dx}(\rho_1 u_1^2) + u_1 \left[\frac{\partial}{\partial y}(\rho v)\right]_1 ,$$

and from (2.82) and (2.19) we have

$$f = 1 - \frac{\frac{d}{dx}(K_m - K_1) - u_1 \left[\frac{\partial(\rho v)}{\partial y}\right]_1}{\mu_0 \left(\frac{\partial u}{\partial y}\right)_0}$$

in which

$$K_m = \int_0^1 \rho u^2 dy, \quad K_1 = \rho_1 u_1^2.$$

If we assume that $\partial(\rho v)/\partial y = 0$ and $dK_m/dx = dK_1/dx$ for the main part of the flow, then f = 1, which is sometimes taken as the basic assumption in calculations on adiabatic flows in pipes [41].

An important feature of f is as follows (this will be used later). The velocity profile is convex along the flow direction ($\partial^2 u/\partial y^2 \le 0$), and $\mu_0 (\partial u/\partial y)_0 > 0$, so we have essentially $f \ge 0$, so f cannot be negative. Similarly we put

$$\mu_1 \left(\frac{\partial^2 \Theta}{\partial y^2}\right)_1 = - F\mu_0 \left(\frac{\partial \Theta}{\partial y}\right)_0 . \tag{2.83}$$

As above, we see that F for $[\partial(\rho v)/\partial y]_1 = 0$ characterizes the change along the length in the difference of the mean heat flux (per second) and that at the axis:

$$F = 1 - \frac{\frac{d}{dx}(Q_m - Q_1) - c_p\Theta_1 \left[\frac{\partial(\rho v)}{\partial x}\right]_1}{-\lambda_0 \left(\frac{\partial \Theta}{\partial y}\right)_0},$$

in which

$$Q_m = c_p \int_0^1 \rho u \Theta dy, \quad Q_1 = c_p \rho_1 u_1 \Theta_1 .$$

Heating alone is considered, so $(\partial\Theta/\partial y)_0 < 0$; for the same reason, the profile of stagnation temperatures must be concave in the flow direction, so $\partial^2\Theta/\partial y^2 \geq 0$, and hence $F \geq 0$ from (2.83).

Substitution of (2.82) and (2.83) into (2.81) gives

$$
\begin{aligned}
N_1 &= fN_1' - N_1'', \\
N_2 &= fN_2' - N_2'', \\
N_3 &= fN_3' - N_3'' \\
N_4 &= FN_4',
\end{aligned}
\qquad (2.84)
$$

in which

$$N_1' = \frac{2k}{k-1}\,\frac{I_1^2}{G_1}\,\mu_0\left(\frac{\partial u}{\partial y}\right)_0,$$

$$N_2' = \frac{\frac{2k}{k-1} I_2 + 1}{G_1}\, I_1 \mu_0 \left(\frac{\partial u}{\partial y}\right)_0,$$

$$N_3' = \frac{2k}{k-1}\,\frac{I_1 I_3}{G_1}\,\mu_0\left(\frac{\partial u}{\partial y}\right)_0,$$

$$N_4' = -\frac{I_1}{G_1}\,\frac{z_3 - z_1^2}{z_1}\,\mu_0\left(\frac{\partial\Theta}{\partial y}\right)_0,$$

$$N_1'' = \frac{\partial I_1}{\partial\Theta_0}\,\frac{d\Theta_0}{dx}\,,$$

$$N_2'' = \frac{I_1}{G_1}\,\mu_0\left(\frac{\partial u}{\partial y}\right)_0 + \frac{\partial I_2}{\partial\Theta_0}\,\frac{d\Theta_0}{dx},$$

$$N_3'' = \frac{I_1}{G}\,\mu_0\left(\frac{\partial\Theta}{\partial y}\right)_0 + \frac{\partial I_3}{\partial\Theta_0}\,\frac{d\Theta_0}{dx}$$

and so, from (2.36) and (2.53),

$$\delta = \left(\frac{158\bar{\mu}_\delta^{1+\frac{1}{\beta}} I_1}{\tilde{R}eG_1 z_1}\right)^{\frac{1}{z_2+1}} \qquad (2.85)$$

All coefficients in (2.79) are now known as functions of z_1, z_2, z_3, z_4, x, f, and F.

The system is integrated as follows.

We assume that we are given the speed $u_1 \equiv z_1 = z_{1i}$ in the initial cross-section ($x = 0$, where $\Theta_1 \equiv z_3 = 1$) and that we know that the velocity has a Blasius profile: $n \equiv z_2 = 1/7$. We also suppose that Reynolds analogy ($m \equiv z_4 = z_2 = 1/7$) for this section.

We now consider whether the analogy and the Blasius profile can apply to the whole flow up to the crisis point (see chapter III for the conditions for onset of crisis).

Here we must put $z_2 = z_4 = 1/7$ and $dz_2/dx = dz_4/dx = 0$ in (2.79), which gives the system the form

$$N_{i1}\frac{dz_1}{dx} + N_{i3}\frac{dz_3}{dx} = N_i \qquad (2.86)$$

$$(i = 1, 2, 3),$$

$$\frac{dz_3}{dx} = N_4.$$

Let us emphasize that the coefficients are functions of z_1, z_3, and x only, for $z_2 = z_4 = 1/7$.

Consider the conditions for compatibility for these four equations in two unknowns.

The first three equations in (2.86) do not contain F; they are compatible if

$$\begin{vmatrix} N_{11} & N_{13} & N_1 \\ N_{21} & N_{23} & N_2 \\ N_{31} & N_{33} & N_3 \end{vmatrix} = 0,$$

whence substitution for the N_i according to (2.84), gives

$$f = \begin{vmatrix} N_{11} & N_{13} & N_1'' \\ N_{21} & N_{23} & N_2'' \\ N_{31} & N_{33} & N_3'' \end{vmatrix} : \begin{vmatrix} N_{11} & N_{13} & N_1' \\ N_{21} & N_{23} & N_2' \\ N_{31} & N_{33} & N_3' \end{vmatrix}$$

We find f and substitute for it in any two of the first three equations in (2.86) to get a system defining z_1 and z_3; the fourth equation gives F as $F = (dz_3/dx)/N_4'$. The behavior of f and F must be considered in the integration.

If f and F are positive up to the crisis point, the Reynolds analogy and the Blasius profile apply up to that point; calculations (chapter IV) show that f becomes zero for a certain $x = x_f$ and then becomes negative (the last if we suppose that $m = n =$ constant). This means that we cannot have $m = n =$ constant for $x \geq x_f$.

There are difficulties in calculating the flow for $x \geq x_f$; chapters III and IV show how this can be done.

11. Method of Calculation for a Given Heat Flux

Here the distribution of the specific heat flux at the wall is given as $q_0 = q_0(x)$, so the wall temperature is one of the unknowns, which we give the symbol $\Theta_0 = z_5$. Here we have five unknowns instead of four, and the system of (2.79) (in which we must put $\Theta_0 = z_5$) must be supplemented with a further equation. This is readily found on the basis of (2.15):

$$-\lambda_0\,(z_5)\left(\frac{\partial\Theta}{\partial y}\right)_0 = q_0. \tag{2.87}$$

$(\partial\Theta/\partial y)_0$ is dependent on the z_i ($i = 1, 2, 3, 4, 5$), so

$$q_0(x) = \Phi(z_1, z_2, z_3, z_4, z_5), \tag{2.88}$$

in which Φ is some function of the z_i.

λ_0 is dependent solely on the wall temperature, i.e. on z_5.

(2.88) is the fifth equation to be added to the system of (2.79); it is more convenient to have it in differential form, because the first three equations in (2.79) contain dz_5/dx.

Differentiation of (2.88) gives

$$\sum_{k=1}^{5} \frac{\partial\Phi}{\partial z_k}\frac{dz_k}{dx} = \frac{dq_0}{dx}. \tag{2.89}$$

The new system of five equations is put in the form

$$\sum_{k=1}^{5} N_{ik}\frac{dz_k}{dx} = N_{iq} \quad (i = 1, 2, 3, 4, 5). \tag{2.90}$$

The N_{ik} ($i, k = 1, 2, 3$ and 4) are defined by (2.80); the N_{i5} are clearly $N_{i5} = \partial I_i/\partial z_5$ ($i = 1, 2, 3$), with $N_{45} = 0$.

The right-hand sides N_{1q}, N_{2q} and N_{3q} differ from those in the equations of (2.79) in not containing terms in $d\Theta_0/dx$; also $N_{4q} = N_4$.

It remains to find the coefficients in the fifth equation in (2.90), namely (2.89), from which we have directly

$$N_{5i} = \frac{\partial\Phi}{\partial z_i} \quad (i = 1, 2, 3, 4, 5),$$

$$N_5 = \frac{dq_0}{dx} .$$

System (2.90) is solved by a method analogous to that given in [10].

Integration of (2.79) or (2.90) presents no essential difficulty if performed with a computer.

Before giving results I consider some aspects that give a deeper understanding of the nature of these effects.

12. Approximate Calculation for the Initial Section

It has so far been assumed that the calculation starts at some channel cross-section where the flow has become stabilized and where (for $Re < 10^5$) the form parameter for the velocity profile is constant (as confirmed by experiment). The Reynolds analogy (also confirmed by experiment) implies that the form parameter for the velocity profile must also remain unchanged in this part. These two parameters are equal for the main part of the flow.

It is of interest to find an estimate for the length of the initial part (i.e., the distance from the inlet to the place at which the flow has become stabilized).

The following method can be used. We assume that m and n differ from their values at the inlet (zero) but cannot be greater than $1/7$ for $Re < 10^5$. If they do become $1/7$, they subsequently remain at that value. The curvature of the two profiles at the axis is taken to be zero until they have reached $1/7$.

The two profiles are taken as being quite flat at the inlet ($m = n = 0$).

We put $f = F = 0$ in (2.79) to get three equations in u_1, m, and n ($\Theta_0 = 1$ in this part, because $F = 0$).

This system is solved up to the point where either $n = 1/7$ or $m = 1/7$.

If at some point $x = x_n$ we have $n = 1/7$, $m < 1/7$, this implies that dynamic stabilization has already set in, so for all later points we must have $n = 1/7$ and must insert f in (2.79). The system is now one of three equations (since $\Theta_1 = 1$ still) in the unknowns m and u_1 and the parameter f. We deduce f from the condition of compatibility to give two equations for u_1 and m. The calculation then proceeds until we reach the point $x = x_{mn}$ at which $m = 1/7$, at which point dynamic and thermal forms of stabilization have been attained; x_{mn} is then the length of the initial part.

If at some point $x = x_m$ we have $m = 1/7$ but $n < 1/7$, we have an analogous calculation, except that F is introduced instead of f.

Of course, the actual effects in the initial part are much more complicated, so this method is only rough and qualitative. A more detailed discussion involves consideration of the possibility of laminar flow at the inlet, which later gives way to turbulent flow.

13. Calculations for $Re > 10^5$

If $Re > 10^5$, the calculations become more complicated, because the form parameter for the velocity profile is found to be variable even in the main part of the tube; it is a function of Re.

The universal logarithmic profile [37] should be used here, but this causes technical difficulties in the calculations.

A simpler method is to use a profile of the type $u_1 y^n$, in which n is a function of Re found by experiment.

14. Flow in Circular Tubes

To assist the practical use of calculations on plane-parallel channels I illustrate how the results can be applied to flow in circular tubes.

The initial system of equations analogous to (2.8)-(2.12) may be put as

$$\rho\left(u\frac{\partial u}{\partial \tilde{x}} + \tilde{R}e_1 v\frac{\partial u}{\partial y}\right) = -\frac{k-1}{2k}\frac{dp}{d\tilde{x}} + \frac{1}{r}\frac{\partial}{\partial r}\left[r(\mu+\epsilon)\frac{\partial u}{\partial r}\right], \quad (2.91)$$

$$\rho\left(u\frac{\partial \Theta}{\partial \tilde{x}} + \tilde{R}e_1 v\frac{\partial \Theta}{\partial y}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left[r(\mu+\epsilon)\frac{\partial \Theta}{\partial r}\right], \quad (2.92)$$

$$\frac{\partial(\rho u r)}{\partial \tilde{x}} + \tilde{R}e_1\frac{\partial(\rho v r)}{\partial y} = 0, \quad (2.93)$$

$$p = \rho T, \quad (2.94)$$

$$\Theta = T + u^2. \quad (2.95)$$

Here r is distance measured perpendicular to the axis of the tube ($r = 1 - y$); $\tilde{R}e_I$ is referred to the radius r_0 of the tube;

$x = r_0 \tilde{Re}_I \tilde{x}$ (r_0 and x are dimensional); and the radius of the tube is the scale for the transverse coordinate.

Integration of (2.93) gives

$$\frac{d}{d\tilde{x}} \int_0^1 \rho u r dy = G. \tag{2.96}$$

Let G be the total gas flow rate; then (2.96) gives

$$2 \int_0^1 \rho u r dy = G. \tag{2.97}$$

We multiply (2.91) by r and integrate with respect to y for the wall to the axis to get

$$\int_0^1 \rho u r \frac{\partial u}{\partial \tilde{x}} dy + \tilde{Re}_1 \int_0^1 \rho v r \frac{\partial u}{\partial y} dy = - \frac{k-1}{2k} \frac{dp}{d\tilde{x}} \int_0^1 r dy + \left[r(\mu+\epsilon) \frac{\partial u}{\partial r} \right]_0^1. \tag{2.98}$$

Now (2.93) gives

$$\tilde{Re}_1 \int_0^1 \rho v r \frac{\partial u}{\partial \tilde{x}} dy = \tilde{Re}_1 [\rho v r n]_1^0 - \tilde{Re}_1 \int_0^1 u \frac{\partial(\rho v r)}{\partial \tilde{x}} dy = \int_0^1 u \frac{\partial(\rho u r)}{\partial \tilde{x}} dy,$$

and $r_1 = 0$, $r_0 = 1$, $\epsilon_0 = 0$, and also

$$\int_0^1 r dy = \int_0^1 (1 - y)\, dy = \frac{1}{2},$$

So we have in place of (2.98) that

$$\frac{d}{d\tilde{x}} 2 \int_0^1 \rho u^2 r dy + \frac{k-1}{2k} \frac{dp}{d\tilde{x}} + 2\mu_0 \left(\frac{\partial u}{\partial y} \right)_0 = 0. \qquad (2.99)$$

Similarly, from (2.92) and (2.93) we have

$$\frac{d}{d\tilde{x}} 2 \int_0^1 \rho u \Theta r dy + 2\mu_0 \left(\frac{\partial \Theta}{\partial y} \right)_0 = 0, \qquad (2.100)$$

with r = 0 (i.e., y = 1) in (2.91) and (2.92).

Now

$$\frac{1}{r} \frac{\partial}{\partial r} \left[r(\mu + \epsilon) \frac{\partial u}{\partial r} \right] = \frac{\mu + \epsilon}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \left(\frac{\partial \mu}{\partial r} + \frac{\partial \epsilon}{\partial r} \right) \frac{\partial u}{\partial r},$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left[r\, (\mu + \epsilon) \frac{\partial \Theta}{\partial r} \right] = \frac{\mu + \epsilon}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Theta}{\partial r} \right) + \left(\frac{\partial \mu}{\partial r} + \frac{\partial \epsilon}{\partial r} \right) \frac{\partial \Theta}{\partial r},$$

so, since $(\partial u/\partial r)_1 = (\partial \Theta/\partial r)_1 = 0$ and $\epsilon_1 = 0$*, and with

*All deductions remain true for $\epsilon_1 \neq 0$.

$$\left[\frac{\mu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \right]_1 = - \tilde{f}\mu_0 \left(\frac{\partial u}{\partial y} \right)_0 ,$$

$$\left[\frac{\mu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Theta}{\partial r} \right) \right]_1 = - \tilde{F}\mu_0 \left(\frac{\partial \Theta}{\partial y} \right)_0 ,$$

we get

$$\rho_1 u_1 \frac{du_1}{d\tilde{x}} + \frac{k-1}{2k} \frac{dp}{d\tilde{x}} + \tilde{f}\mu_0 \left(\frac{\partial u}{\partial y} \right)_0 = 0, \qquad (2.101)$$

$$\rho_1 u_1 \frac{d\Theta_1}{d\tilde{x}} + \tilde{F}\mu_0 \left(\frac{\partial \Theta}{\partial y} \right)_0 = 0. \qquad (2.102)$$

(2.96) and (2.99)-(2.102) form a system that defines the flow parameters.

The velocity and temperature profiles are given as

$$u = u_1 \Omega^n , \qquad (2.103)$$

$$\Theta - \Theta_0 = (\Theta_1 - \Theta_0)\, \Omega^m , \qquad (2.104)$$

in which $\Omega = 1 - r^2$.

It is readily seen that the integrals appearing in (2.90), (2.97), and (2.100) may be expressed in terms of the integrals I_1, I_2, I_3, which have been deduced in this chapter.

In fact

$$2\int_0^1 \rho u r dy = 2\int_0^1 \rho u r dr = p\int_0^1 \frac{u}{\Theta - u^2}\, d\Omega = pI_1$$

and similarly

$$2\int_0^1 \rho u^2 r dy = pI_2,$$

$$2\int_0^1 \rho u \Theta r dy = pI_3.$$

This feature enables us to put (2.96), (2.99), and (2.100) as

$$\frac{d}{d\tilde{x}}(pI_1) = 0, \tag{2.105}$$

$$\frac{d}{d\tilde{x}}(pI_2) + \frac{k-1}{2k}\frac{dp}{d\tilde{x}} + 2\mu_0\left(\frac{\partial u}{\partial y}\right)_0 = 0, \tag{2.106}$$

$$\frac{d}{d\tilde{x}}(pI_3) + 2\mu_0\left(\frac{\partial \Theta}{\partial y}\right)_0 = 0. \tag{2.107}$$

Further, (2.97) gives

$$pI_1 = G. \tag{2.108}$$

We replace y by Ω in (2.101), (2.102), and (2.105)-(2.107) to get

$$\left(\frac{\partial}{\partial y}\right)_0 = \left(\frac{\partial}{\partial\Omega}\right)_0 \left(\frac{d\Omega}{dy}\right)_0 = 2\left(\frac{\partial}{\partial\Omega}\right)_0.$$

If now we put

$$\bar{x} = 4\tilde{x} = 4r_0\tilde{Re}_1 x, \quad \tilde{f} = 2f, \quad \tilde{F} = 2F,$$

we have a system identical with that considered in this chapter for plane-parallel flow.

Here, as in section 3, we readily find that

$$\zeta = \frac{8\mu_0}{\tilde{Re}_1 G u_m}\left(\frac{\partial u}{\partial y}\right)_0.$$

As in section 6, we find that

$$St = -\frac{1}{\tilde{Re}_1 G}\,\frac{\mu_0}{\Theta_0 - \Theta_m}\left(\frac{\partial\Theta}{\partial y}\right)_0.$$

We determine $\mu_0\,(\partial u/\partial y)_0$ and $\mu_0\,(\partial\Theta/\partial y)_0$ from (2.31) and (2.36); with (2.103) and (2.104) we get

$$\mu_0\left(\frac{\partial u}{\partial y}\right)_0 = \bar{\mu}_\delta\,\frac{\mu_\delta}{\delta} = \frac{u_1\Omega_\delta^n}{\delta}\,\bar{\mu}_\delta,$$

$$\mu_0 \left(\frac{\partial\Theta}{\partial y}\right)_0 = \bar{\mu}_\delta \frac{\Theta_\delta - \Theta_0}{\delta} = \frac{(\Theta_1 - \Theta_0)\Omega_\delta^m}{\delta} \bar{\mu}_\delta .$$

But $\Omega = 1 - r^2 = 1 - (1 - y^2) = 2y(1 - y)$, so we have $\Omega = 2$ if we neglect δ relative to one, and then

$$\mu_0 \left(\frac{\partial u}{\partial y}\right)_0 = 2u_1 \Omega_\delta^{n-1} \bar{\mu}_\delta , \quad \mu_0 \left(\frac{\partial\Theta}{\partial y}\right)_0 = 2 (\Theta_1 - \Theta_0)\Omega_\delta^{m-1} \bar{\mu}_\delta .$$

It remains to find Ω_δ in order to find the resistance and heat-transfer coefficients. This is readily done from

$$\frac{u_\delta \delta}{\bar{\nu}_\delta} = Re_\delta ,$$

hence

$$\Omega_\delta = \left(\frac{2\bar{\nu}_\delta Re_\delta}{u_1}\right)^{\frac{1}{n+1}}$$

Then every result obtained for the power profiles (powers m and n) for a plane-parallel channel can be transferred to a circular tube having profiles in powers of the variable Ω, provided the length scales are suitably adjusted.

It is assumed that the diameter of the tube is sufficiently large for its curvature not to alter the results substantially.

Note that the flow calculations for a tube can also be performed with power profiles, but they are then more cumbrous.

CHAPTER III

GAS FLOW IN A SYMMETRICALLY HEATED CHANNEL WITH PLANE-PARALLEL WALLS: QUALITATIVE ANALYSIS

Relationships simpler than those in chapter II are derived here, which enable us to deduce the characteristic features without resort to computations.

Dorodnitsyn's variable [8] enables us to simplify the equations for the flow of a compressible fluid considerably. This variable here gives us a system of equations analogous to (2.79) or (2.90), but with very simple expressions for the coefficients.

We put

$$\eta = \int_0^y \rho dy, \quad y = \int_0^\eta \frac{d\eta}{\rho} \tag{3.1}$$

and

$$H = \frac{\eta}{\eta_1}, \quad \eta_1 = \int_0^1 \rho dy, \tag{3.2}$$

on the assumption that the profiles of velocity and stagnation temperature can be put as

$$u = u_1 H^n, \tag{3.3}$$

$$\Theta - \Theta_0 = (\Theta_1 - \Theta_0)\, H^m. \tag{3.4}$$

The symbols for the form parameters m and n are as in chapter II, which cannot lead to confusion, because only the profiles of (3.3) and (3.4) are considered here.

The relation of y to H is given by (3.1) and (3.2) as

$$y = \int_0^{\eta} \frac{d\eta}{\rho} = \eta_1 \int_0^{H} \frac{dH}{\rho},$$

whence with the density

$$\rho = \frac{p}{T} = \frac{p}{\Theta - u^2}$$

we have

$$y = \frac{\eta_1}{p} \int_0^{H} (\Theta - u^2)\, dH. \tag{3.5}$$

The powers of H in (3.3) and (3.4) do not remain simple power functions in the physical plane (i.e., in y), so all the results given here are somewhat qualitative; but they are of interest as indicating the nature of the effects.

1. System of Equations

For convenience we rewrite (2.17), (2.19), (2.20), (2.27), and (2.28) as

$$\int_0^1 u\rho dy = G_1,$$

$$\frac{d}{dx}\int_0^1 u^2\rho dy + \frac{k-1}{2k}\frac{dp}{dx} + \mu_0\left(\frac{\partial u}{\partial y}\right)_0 = 0,$$

$$\frac{d}{dx}\int_0^1 u\Theta\rho dy + \mu_0\left(\frac{\partial \Theta}{\partial y}\right)_0 = 0, \tag{3.6}$$

$$\rho_1 u_1\frac{du_1}{dx} + \frac{k-1}{2k}\frac{dp}{dx} = \mu_1\left(\frac{\partial^2 u}{\partial y^2}\right)_1,$$

$$\rho_1 u_1\frac{d\Theta_1}{dx} = \mu_1\left(\frac{\partial^2 \Theta}{\partial y^2}\right)_1.$$

As in chapter II, we assume that (3.3) and (3.4) are applicable right out to the wall in the computation of the integrals appearing in (3.6).

Now

$$\int_0^1 \Phi\ (y)\ \rho dy = \eta_1 \int_0^1 \Phi\ (y\ (H)\ )\ dH,$$

in which Φ is an arbitrary function; then with

$$I_u = \int_0^1 udH = \frac{u_1}{n+1}, \tag{3.7}$$

$$I_{uu} = \int_0^1 u^2 dH = \frac{u_1^2}{2n + 1}, \qquad (3.8)$$

$$I_\Theta = \int_0^1 \Theta dH = \frac{m\Theta_0 + \Theta_1}{m + 1}, \qquad (3.9)$$

$$I_{u\Theta} = \int_0^1 \Theta u dH = \frac{u_1}{n + 1} \frac{m\Theta_0 + (n + 1)\Theta_1}{m + n + 1} \qquad (3.10)$$

and with, as in chapter II,

$$\mu_1 \left(\frac{\partial^2 u}{\partial y^2}\right)_1 = - f\mu_0 \left(\frac{\partial u}{\partial y}\right)_0,$$

$$\mu_1 \left(\frac{\partial^2 \Theta}{\partial y^2}\right)_1 = - F\mu_0 \left(\frac{\partial \Theta}{\partial y}\right)_0,$$

we have in place of (3.6), by use of the equation of state,

$$\eta_1 I_u = G_1 \qquad (3.11)$$

$$\frac{d}{dx} (\eta_1 I_{uu}) + \frac{k-1}{2k} \frac{dp}{dx} + \mu_0 \left(\frac{\partial u}{\partial y}\right)_0 = 0, \qquad (3.12)$$

$$\frac{d}{dx}(\eta_1 I_{u\Theta}) + \mu_0\left(\frac{\partial\Theta}{\partial y}\right)_0 = 0, \tag{3.13}$$

$$\frac{pu_1}{\Theta_1 - u_1^2}\frac{du_1}{dx} + \frac{k-1}{2k}\frac{dp}{dx} + f\mu_0\left(\frac{\partial u}{\partial y}\right)_0 = 0, \tag{3.14}$$

$$\frac{pu_1}{\Theta_1 - u_1^2}\frac{d\Theta_1}{dx} + F\mu_0\left(\frac{\partial\Theta}{\partial y}\right)_0 = 0. \tag{3.15}$$

We also put $y = H = 1$ in (3.5) to get a relation between pressure p and the variable η on the axis η_1:

$$p = \eta_1 (I_\Theta - I_{uu}). \tag{3.16}$$

From (3.11) we have

$$\frac{1}{\eta_1}\frac{d\eta_1}{dx} = -\frac{1}{I_u}\frac{dI_u}{dx}, \tag{3.17}$$

so from (3.12) and (3.14) we find by use of (3.11) and (3.16) that

$$I_{uu}\left(\frac{1}{I_{uu}}\frac{dI_{uu}}{dx} - \frac{1}{I_u}\frac{dI_u}{dx}\right) - \frac{I_\Theta - I_{uu}}{\Theta_1 - u_1^2}u_1\frac{du_1}{dx} = \frac{I_u}{G_1}\mu_0\left(\frac{\partial u}{\partial y}\right)_0 (f-1). \tag{3.18}$$

From (3.14) with (3.11), (3.16), and (3.17) we have

$$\frac{2k}{k-1}\frac{u_1}{\Theta_1 - u_1^2}\frac{du_1}{dx} - \frac{1}{I_u}\frac{dI_u}{dx} + \frac{1}{I_\Theta - I_{uu}}\left(\frac{dI_\Theta}{dx} - \frac{dI_{uu}}{dx}\right) =$$

$$= - \frac{2k}{k-1}\frac{I_u}{G_1}(I_\Theta - I_{uu})\,\mu_0\left(\frac{\partial u}{\partial y}\right)_0 f. \tag{3.19}$$

From (3.13) we have

$$I_{u\Theta}\left(\frac{1}{I_{u\Theta}}\frac{dI_{u\Theta}}{dx} - \frac{1}{I_u}\frac{dI_u}{dx}\right) = - \frac{I_u}{G_1}\mu_0\left(\frac{\partial\Theta}{\partial y}\right)_0. \tag{3.20}$$

From (3.15)

$$\frac{d\Theta_1}{dx} = - \frac{\Theta_1 - u_1^2}{u_1(I_\Theta - I_{uu})}\frac{I_u}{G_1}\left(\frac{\partial\Theta}{\partial y}\right)_0 F. \tag{3.21}$$

The system of (3.18)-(3.21) may be put in the form

$$Z_{i1}\frac{du_1}{dx} + Z_{i2}\frac{dn}{dx} + Z_{i3}\frac{d\Theta_1}{dx} + Z_{i4}\frac{dm}{dx} = Z_i \tag{3.22}$$

$$(i = 1, 2, 3, 4).$$

The coefficients of (3.22) are readily found by the use of (3.7)-(3.10). The following table gives the general expressions for these and also the results after the integrals (3.7)-(3.10) and their derivatives have been inserted (the final expressions). The expressions have been simplified by the use of the symbol

Coefficients in the System of Equations of (3.22)

Symbol	General Expression	Final Expression
Z_{11}	$I_{uu}\left(\frac{1}{I_{uu}}\frac{\partial I_{uu}}{\partial u_1} - \frac{1}{I_u}\frac{\partial I_u}{\partial u_1}\right) - u_1(I_\Theta - I_{uu})$	$\frac{u_1}{2n+1}\,\frac{\Theta_1 - 1}{\Theta_1 - u_1^2}$
Z_{12}	$I_{uu}\left(\frac{1}{I_{uu}}\frac{\partial I_{uu}}{\partial n} - \frac{1}{I_u}\frac{\partial I_u}{\partial n}\right)$	$-\frac{u_1^2}{(2n+1)^2(n+1)}$
Z_{13}	0	0
Z_{14}	0	0
Z_{21}	$\frac{2k}{k-1}\,\frac{u_1}{\Theta_1 - u_1^2} - \frac{1}{I_u}\frac{\partial I_u}{\partial u_1} - \frac{1}{I_\Theta - I_{uu}}\frac{\partial I_{uu}}{\partial u_1}$	$\frac{2k}{k-1}\,\frac{u_1}{\Theta_1 - u_1^2} - \frac{1}{u_1}\,\frac{I+u_1^2}{I-u_1^2}$
Z_{22}	$-\frac{1}{I_u}\frac{\partial I_u}{\partial n} - \frac{1}{I_\Theta - I_{uu}}\frac{\partial I_{uu}}{\partial n}$	$\frac{(2n+1)I+u_1^2}{(n+1)(2n+1)(I-u_1^2)}$
Z_{23}	$\frac{1}{I_\Theta - I_{uu}}\,\frac{\partial I_\Theta}{\partial \Theta_1}$	$\frac{2n+1}{m+1}\,\frac{1}{I-u_1^2}$

Coefficients in the System of Equations of (3.22)

Symbol	General Expression	Final Expression
Z_{24}	$\frac{1}{I_\Theta - I_{uu}} \frac{\partial I_\Theta}{\partial m}$	$\frac{2n+1}{(m+1)^2} \frac{\Theta_0-\Theta_1}{I-u_1^2}$
Z_{31}	$I_{u\Theta}\left(\frac{1}{I_{u\Theta}} \frac{\partial I_{u\Theta}}{\partial u_1} - \frac{1}{I_u} \frac{\partial I_u}{\partial u_1}\right)$	0
Z_{32}	$I_{u\Theta}\left(\frac{1}{I_{u\Theta}} \frac{\partial I_{u\Theta}}{\partial u_1} - \frac{1}{I_u} \frac{\partial I_u}{\partial u_1}\right)$	$-\frac{m}{n+1} \frac{\Theta_0-\Theta_1}{(m+n+1)^2} u_1$
Z_{33}	$\frac{\partial I_{u\Theta}}{\partial \Theta_1}$	$\frac{u_1}{m+n+1}$
Z_{34}	$\frac{\partial I_{u\Theta}}{\partial m}$	$\frac{\Theta_0-\Theta_1}{(m+n+1)^2} u_1$
Z_{41}	0	0
Z_{42}	0	0
Z_{43}	1	1
Z_{44}	0	0

$$I = (2n + 1)\, I_\Theta = \frac{2n + 1}{m + 1}\,(m\Theta_0 + \Theta_1). \tag{3.23}$$

The right-hand sides of (3.22) are

$$Z_1 = \frac{u_1}{n + 1}\,\frac{\mu_0}{G_1}\left(\frac{\partial u}{\partial y}\right)_0 (f - 1),$$

$$Z_2 = -\frac{2k}{k - 1}\,\frac{u_1}{n + 1}\,\frac{\mu_0}{G_1}\,\frac{I - u_1^2}{2n + 1}\left(\frac{\partial u}{\partial y}\right)_0 f - \frac{2n + 1}{I - u_1^2}\,\frac{m}{m + 1}\,\frac{d\Theta_0}{dx},$$

$$Z_3 = -\frac{u_1}{n + 1}\,\frac{\mu_0}{G_1}\left(\frac{\partial \Theta}{\partial y}\right)_0 - \frac{u_1}{n + 1}\,\frac{m}{m + n + 1}\,\frac{d\Theta_0}{dx},$$

$$Z_4 = \frac{2n + 1}{n + 1}\,\frac{\Theta_1 - u_1^2}{I - u_1^2}\,\frac{F\mu_0}{G_1}\left(\frac{\partial \Theta}{\partial y}\right)_0.$$

A qualitative analysis is possible once the coefficients of (3.22) have been determined. First I consider the crisis effect. In order to include more complicated cases I start with one-dimensional flow, then consider adiabatic flow, and finally the general case.

2. Crisis in a One-Dimensional Flow

In this case ρ, u, and Θ are constant in any given cross-section; the last two equations in (3.6) become meaningless, while the first three reduce to one-dimensional equations of chapter I:

$$\rho u = G_1,$$

$$\rho u \frac{du}{dx} + \frac{k-1}{2k}\frac{dp}{dx} = -\bar{\zeta}, \qquad (3.24)$$

$$\frac{d\Theta}{dx} = \bar{q}_0,$$

in which $\bar{\zeta}$ and $\bar{q}_0$ denote the resistance coefficient and heat flux together with some positive factors.

To these three we add the equation of state

$$\rho = \frac{p}{T} = \frac{p}{\Theta - u^2}, \qquad (3.25)$$

which, with the first equation of (3.24), can be put as

$$\frac{pu}{\Theta - u^2} = G_1,$$

so

$$\frac{1}{p}\frac{dp}{dx} = -\frac{1}{u}\frac{du}{dx} + \frac{1}{\Theta - u^2}\left(\frac{d\Theta}{dx} - 2u\frac{du}{dx}\right). \qquad (3.26)$$

We substitute for dp/dx in the second equation of (3.24) and use (3.25) to get a system in u and Θ:

$$\left(\frac{k+1}{k-1} - \frac{\Theta}{u^2}\right)\frac{du}{dx} + \frac{1}{u}\frac{d\Theta}{dx} = -\frac{2k}{k-1}\frac{\bar{\zeta}}{\rho u}, \tag{3.27}$$

$$\frac{d\Theta}{dx} = \bar{q}_0.$$

We solve this for du/dx and $d\Theta/dx$ to get

$$\frac{du}{dx} = \frac{\Delta_1}{\Delta}, \quad \frac{d\Theta}{dx} = \frac{\Delta_2}{\Delta}, \tag{3.28}$$

in which

$$\Delta = \begin{vmatrix} \frac{k+1}{k-1} - \frac{\Theta}{u^2} & \frac{1}{u} \\ 0 & 1 \end{vmatrix} = \frac{k+1}{k-1} - \frac{\Theta}{u^2} \; ;$$

$$\Delta_1 = \begin{vmatrix} -\frac{2k}{k-1}\frac{\bar{\zeta}}{\rho u} & \frac{1}{u} \\ \bar{q}_0 & 1 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} \frac{k+1}{k-1} - \frac{\Theta}{u^2} & -\frac{2k}{k+1}\frac{\bar{\zeta}}{\rho u} \\ 0 & \bar{q}_0 \end{vmatrix}$$

Let $\Delta < 0$ in the initial cross section, which corresponds to

$$V^2 = \frac{u^2}{\Theta} < \frac{k - 1}{k + 1},$$

(local subsonic speed).

With $\bar{q}_0 \geq 0$ (i.e., assuming that the flow is adiabatic or that there is an influx of heat) we have $\Delta_1 < 0$, so $du/dx > 0$, and so u increases.

This increase is the more rapid the smaller the absolute value of Δ. If Θ and u are continuous functions of x, Δ cannot become positive without passing through zero.

But for $\Delta \to -0$ we have, because $\Delta_1 < 0$, that $du/dx \to +\infty$, so any solution to (3.28) then becomes meaningless. If the flow could continue in this way, we would have $\Delta > 0$, so $du/dx < 0$. But Θ increases, so $\Delta = (k + 1)/(k - 1) - \Theta/u^2$ must decrease, i.e. Δ cannot become positive, which proves our assertion.

The flow here is thus possible only up to speeds not exceeding the speed of sound.

The condition for onset of crisis is very simple for the case of one-dimensional flow: $V^2 = (k - 1)/(k + 1)$, so the crisis occurs when the speed equals the local speed of sound.

The number of variables increases if the flow is in more dimensions (the form parameters for the velocity and temperature profiles appear), so the condition of onset becomes more complicated.

A necessary condition here is still $\Delta = 0$, in which Δ is the determinant of the system. It is here much more difficult to decide whether crisis actually occurs. I omit the cumbrous expressions and assume that crisis occurs for $q_0 \geq 0$, which is justified by the one-dimensional theory and by the experimental evidence.

3. Crisis in an Adiabatic Flow

Here $m = 0$, $\Theta_0 = \Theta_1 = 1$, and (3.22) becomes

$$Z_{11}\frac{du_1}{dx} + Z_{12}\frac{dn}{dx} = Z_1, \qquad (3.29)$$

$$Z_{21}\frac{du_1}{dx} + Z_{22}\frac{dn}{dx} = Z_2.$$

From (3.23) we have $I = 2n + 1$, while from Table 1 we have the coefficients as

$$Z_{11} = -\frac{2n}{2n+1}\frac{u_1}{1-u_1^2},$$

$$Z_{12} = -\frac{u_1^2}{(2n+1)^2(n+1)},$$

$$Z_{21} = \frac{2k}{k-1}\frac{u_1}{1-u_1^2} - \frac{1}{u_1}\frac{2n+1+u_1^2}{2n+1-u_1^2},$$

$$Z_{22} = \frac{(2n+1)^2 + u_1^2}{(n+1)(2n+1)(2n+1-u_1^2)},$$

$$Z_1 = \frac{u_1}{n+1} \frac{\mu_0}{G_1} \left(\frac{\partial u}{\partial y}\right)_0 (f - 1),$$

$$Z_2 = -\frac{2k}{k-1} \frac{u_1}{n+1} \frac{\mu_0}{G_1} \frac{2n+1}{2n+1-u_1^2} \left(\frac{\partial u}{\partial y}\right)_0 f.$$

The relation of u_1 to n at the moment of crisis (if this occurs) is found by putting $\triangle = 0$, with

$$\triangle = \begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix}.$$

Then $\triangle = 0$ is found by substitution and a little manipulation as

$$\frac{2k}{k-1} u_1^2 (2n + 1 - u_1^2) - (2n + 1 + u_1^2)(1 - u_1^2) - 2n\,[(2n + 1)^2 + u_1^2] = 0,$$

or

$$u_1^4 - 2Bu_1^2 + C = 0, \tag{3.30}$$

in which

$$B = \frac{k}{k+1} (2n + 1),$$

$$C = \frac{k-1}{k+1} (2n + 1)\,[1 + 2n(2n + 1)].$$

From (3.30) we have

$$u_1^2 = B \pm \sqrt{B^2 - C}. \tag{3.31}$$

We find the appropriate sign for the root by considering the case n = 0 (one-dimensional flow). From (3.31) we have for this case that

$$u_1^2 = \frac{k}{k+1} \pm \sqrt{\left(\frac{k}{k+1}\right)^2 - \frac{k-1}{k+1}} = \frac{k \pm 1}{k+1}.$$

which shows that we must take the minus sign in (3.31).

Figure 12 shows the relation of u_1 to n for the crisis point (curve 1); only for n = 0 is the speed at the axis equal to the speed of sound at the moment of crisis ($u_1 = [(k - 1)/(k + 1)]^{1/2} = 0.408$).

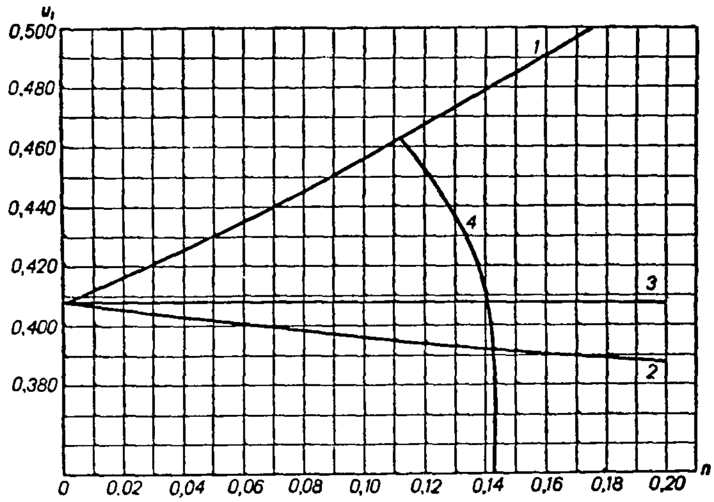
Then $u_1 > [(k - 1)/(k + 1)]^{1/2}$ for $n \neq 0$, so the speed at the axis exceeds the speed of sound at the crisis point.

The flow for the whole channel must be computed before one can say whether a given combination of n and u_1 actually occurs at the crisis point.

Before I give an example I shall show that n cannot be constant near the crisis point.

As in chapter II, I assume that n is constant at n_0 (the value for x = 0) over some regions beyond the initial section. Then because $dn/dx = 0$ we have

$$\Delta_2 = \begin{vmatrix} Z_{11} & Z_1 \\ Z_{21} & Z_2 \end{vmatrix} = 0.$$



[Commas represent decimal points]

Fig. 12.

Relation of u_1 (speed at axis) to n (form parameter) near the crisis point: 1) at crisis point; 2) at point where n begins to change; 3) speed of sound; 4) variation of form parameter in part preceding crisis.

Substitution for the elements and simple manipulation give

$$\Delta_2 = \frac{u_1}{n + 1} \frac{1}{G_1} \left(\frac{\partial u}{\partial y} \right)_0 \left[\frac{2k}{k - 1} \frac{u_1}{1 - u_1^2} \frac{2n(2n + 1 - u_1^2)}{(2n + 1)^2} f - \right.$$

$$- (f - 1)\left(\frac{2k}{k - 1}\frac{u_1}{1 - u_1^2} - \frac{1}{u_1}\frac{2n + 1 + u_1^2}{2n + 1 - u_1^2}\right)\Big], \qquad (3.32)$$

so with $\triangle_2 = 0$ we have

$$f = \frac{\dfrac{2n + 1 + u_1^2}{2n + 1 + u_1^2} - \dfrac{2k}{k - 1}\dfrac{u_1^2}{1 - u_1^2}}{\dfrac{2k}{k - 1}\dfrac{u_1^2}{1 - u_1^2}\dfrac{2n(2n + 1 - u_1^2)}{(2n + 1)^2} \quad \dfrac{2n + 1 + u_1^2}{2n + 1 - u_1^2} \quad \dfrac{2k}{k - 1}\dfrac{u_1^2}{1 - u_1^2}}$$

This f is close to one for u_1 small; f decreases as u_1 increases if n is constant, and it finally becomes negative when $u_1 = u_{1f}$. This means that $\triangle_2 = 0$ is possible only until $f = 0$, i.e.

$$\frac{2k}{k - 1}\frac{u_1}{1 - u_1^2} = \frac{1}{u_1}\frac{2n + 1 + u_1^2}{2n + 1 - u_1^2} . \qquad (3.33)$$

Let x_f be the x at which $f = 0$; if $x_c > x_f$, we assume that $f = 0$ everywhere in the range $[x_f, x_c]$, and in this range we do not have $\triangle_2 = 0$, so n cannot be constant from x_f onwards.

The relation of u_1 to n for $x = x_f$ is given by (3.33) or

$$u_1^4 - \frac{2k}{k+1}\left[\frac{3k-1}{k} n + 1\right] u_1^2 + \frac{k-1}{k+1}(2n+1) = 0.$$

Figure 12 shows that any value of the form parameter differing from zero can remain constant only up to certain speeds at the axis, which are always less than the speed of sound. Only in the one-dimensional case ($n = 0$) can the parameter remain constant up to the speed of sound.

This indicates that crisis occurs for $n > 0$ only after n has begun to vary, i.e., in the part where $f = 0$.

In the determinant

$$\Delta_1 = \begin{vmatrix} Z_1 & Z_{12} \\ Z_2 & Z_{22} \end{vmatrix}$$

we put $f = 0$ to get

$$\Delta_1 = - \frac{u_1}{(n+1)^2(2n+1)} \frac{\mu_0}{G_1} \left(\frac{\partial u}{\partial y}\right)_0 \frac{(2n+1)^2 + u_1^2}{2n + 1 - u_1^2}, \qquad (3.34)$$

which implies that $\Delta_1 < 0$, so crisis does occur, and then $du/dx \to +\infty$ because $\Delta < 0$.

The variation in n between curves 2 and 1 of Fig. 12 may be found from the relation of n to u_1 in this range. The simplest quantity to find is

$$\frac{dn}{du_1} = \frac{dn}{dx} : \frac{du_1}{dx} = \frac{\Delta_2}{\Delta_1}.$$

We put f = 0 in (3.32) and divide by (3.34) to get

$$\frac{dn}{du_1} = - \frac{(n + 1)(2n + 1)}{u_1(1 - u_1^2)} \frac{\left(\frac{3k - 1}{k - 1} 2n + \frac{2k}{k - 1}\right) u_1^2 - \frac{k + 1}{k - 1} u_1^4 - 2n - 1}{(2n + 1)^2 + u_1^2} \tag{3.35}$$

Figure 12 (curve 4) shows results from numerical integration; $n_0 = 1/7 \approx 0.143$ (up to curve 2) has been used here.

Figure 12 indicates that n decreases beyond curve 2, which means that the profile flattens out in the precrisis section.

dn/du_1 tends to a finite limit as the crisis point is approached, as is easily shown from the fact that the second fraction on the right in (3.35) becomes 2nd at the crisis point [this relation of u_1 to n derives from (3.31)]. Then for $x \to x_c$ we have

$$\frac{dn}{du_1} \to - \frac{(n + 1)(2n + 1)}{u_1(1 - u_1^2)} 2n. \tag{3.36}$$

This means that in all cases where it is possible it is much more convenient to perform calculations near the crisis point by taking as our independent variable u_1, not the longitudinal coordinate.

For $x \to x_c$ we have $du_1/dx \to + \infty$, while (3.36) implies that $dn/dx) = (dn/du_1)\,(du_1/dx) \to - \infty$ at this point.

Consider the mean speed at the crisis point. We have from section 1 that

$$u_m = \frac{\int_0^1 \rho u^2 dy}{\int_0^1 \rho u dy} = \frac{\eta_1 I_{uu}}{\eta_1 I_u} = \frac{n+1}{2n+1} u_1 . \qquad (3.37)$$

Figure 13 shows the relation of u_m to n at this point; only for n = 0 is u_m equal to the speed of sound, while for n > 0 it is always greater than that speed.

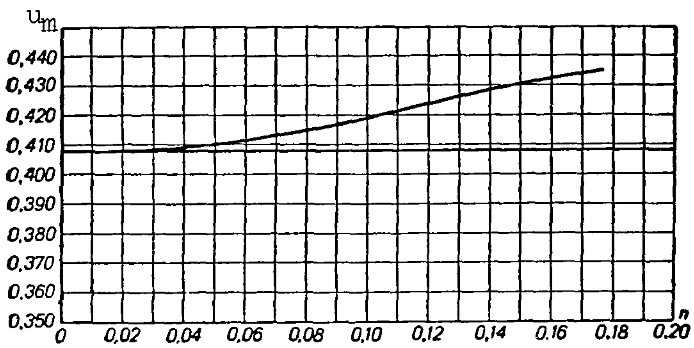


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Fig. 13.

Mean velocity u_m as a function of form parameter n at the crisis point; the straight line corresponds to the speed of sound.

The results may be summarized as follows.

Adiabatic flow in a plane-parallel channel is possible with a constant velocity-profile form parameter n only up to certain values of the speed at the axis, which are less than the speed of sound.

Past this point we have a precrisis section, in which the form parameter must decrease (the flow is accompanied by flattening of the velocity profile). The crisis sets in when the speed at the axis and the mean speed exceed the speed of sound; $du_1/dx \to +\infty$ and $dn/dx \to -\infty$ simultaneously as $x \to x_c$.

4. Adiabatic Flow: Behavior of a Stream Line Near the Axis of the Channel

The possibility of transition through the speed of sound near the axis is a consequence of the special conditions that arise in the precrisis part. These conditions can be found by considering the behavior of a stream line near the axis.

Let $y = y(u_1^2)$ be the equation of a stream line in coordinates y and u_1^2 (Fig. 14). The flow rate between two such lines (in particular, between a line and the wall) must remain constant:

$$\int_0^y \rho u dy = G_y = \text{const},$$

or

$$\int_0^\eta u d\eta = \eta_1 \int_0^H u dH = \eta_1 \frac{u_1}{n+1} H^{n+1} = G_y = \text{const}.$$

But from (3.7) and (3.11) we have $\eta_1 u_1/(n+1) = G$, so

$$H = \left(\frac{G_y}{G_1}\right)^{\frac{1}{n+1}} \tag{3.38}$$

H on a stream line is thus constant while n is constant, and decrease in n is accompanied by decrease in H.

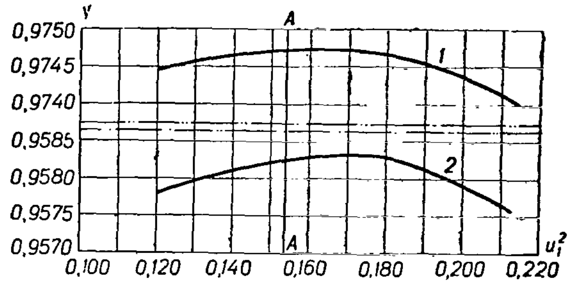

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Fig. 14.

Behavior of stream lines near the axis of a channel. The precrisis section starts at AA. G_y/G_1 as follows:

1) 0.97; 2) 0.95.

(3.5) gives the relation of y to H, is we put $\Theta = 1$ (since adiabatic flow is envisaged):

$$y = \frac{\eta_1}{p}\int_0^H (1 - u^2)\, dH.$$

From (3.8), (3.9), and (3.16) we have

$$y = \frac{2n + 1}{2n + 1 - u_1^2}\left(H - u_1^2 \frac{H^{2n+1}}{2n + 1} \right),$$

or

$$y = H \frac{2n + 1 - H^{2n} u_1^2}{2n + 1 - u_1^2} . \tag{3.39}$$

This last equation shows that y increases with u_1 when H is constant (i.e., n is constant), because $H < 1$; the lines converge on the axis. In the precrisis part they start to diverge slowly (Fig. 14), which produces the conditions for the transition through the speed of sound near the axis not far from the reversal of gradient in the stream lines (see chapter I).

5. Physical Meaning of the Crisis

This can be elucidated by considering the behavior of the stream lines throughout the channel.

Subsonic and supersonic speeds both occur in the precrisis part. The equation for the line of transition through the speed of sound is found as follows. The speed of sound corresponds to $u = [(k - 1)/(k + 1)]^{1/2}$, so on this line we must have

$$\sqrt{\frac{k - 1}{k + 1}} = u_1 H^n ,$$

hence

$$H = \left(\frac{1}{u_1} \sqrt{\frac{k - 1}{k + 1}} \right)^{\frac{1}{n}} \tag{3.40}$$

in which u_1 and n are related by (3.35), which has been integrated above (Fig. 12, curve 3).

We substitute for H from (3.39) and (3.40) to get the relation of y to u_1. Figure 15 shows the transition line OK, which indicates that the transition occurs first at the axis (section OA) and then spreads outwards to the wall. O'K is the crisis section.

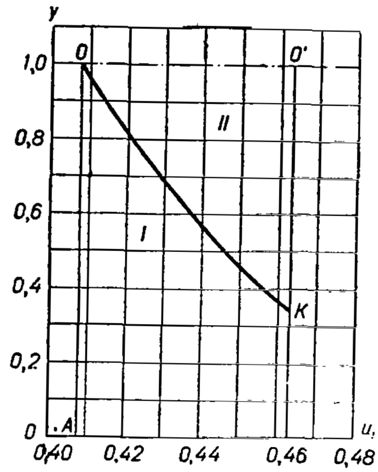

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Fig. 15.

Line of transition through the speed of sound: I) subsonic region; II) supersonic region; OK) transition line.

Consider the behavior of a stream line close to that passing through K.

Subscript K is used with the quantities here, and (3.39) is written for this point to find H_K. Then (3.38), with $n = n_K$, gives us $(G_y/G_1)_K$ for this line through K; we give G_y/G_1 values close to this to find from (3.39) a series of lines close to that passing through K.

Figure 16 shows the results; the stream lines diverge near K, the degree of divergence scarcely varying from line to line.

The resistance increases as the wall is approached, so it is clear that a current tube near K shows the following effects. The resistance and degree of divergence in tube 1-2 are such that the speed can pass through that of sound, whereas an elementary tube enclosing K has a relation of resistance to degree of divergence such that this transition cannot occur; local crisis occurs in this tube.

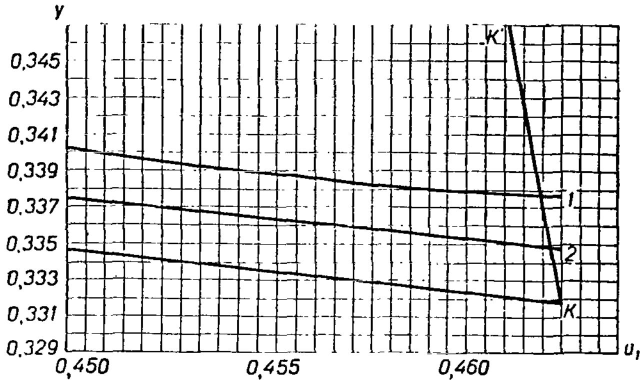

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Fig. 16.

Physical meaning of crisis: KK' transition line; 1 and 2 stream lines.

The crisis is thus due to the impossibility of transition through the speed of sound for one of the stream lines.

6. Resistance Coefficient for Adiabatic Flow

In section 6 of chapter II we derived (2.37) for the resistance coefficient, which for an adiabatic flow may be put in the form*

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*It is assumed that $\mu_\delta \approx 1$.

$$\zeta = \frac{8u_1}{\tilde{Re}G_1 u_m} \delta^{n-1} \tag{3.41}$$

Let $H|_{y=\delta} = H_\delta$; we put $y = \delta$ in (3.1) and neglect u^2 as being small relative to one to get

$$\eta_\delta = \int_0^\delta \rho dy = p \int_0^\delta \frac{dy}{1 - u^2} \approx p\delta,$$

so

$$H = \frac{p}{\eta_1} \delta.$$

We find p/η_1 as in the deduction of (3.39) to get

$$H_\delta = \frac{2n + 1 - u_1^2}{2n + 1} \delta.$$

(2.32) gives us δ as

$$\tilde{Re} \frac{u_\delta \delta}{\nu_0} = Re\delta = 157,$$

so, with* $\nu_0 = \mu_0/\rho_0 = 1/p$, we have

---

*$\beta = 1$ is the assumption in (2.33).

$$\widetilde{Re}\, u_1 \left(\frac{2n + 1 - u_1^2}{2n + 1}\right)^n \delta^{n+1}\, p = 157. \qquad (3.42)$$

We find p from the equations used in deducing (3.39), namely

$$p = \eta_1 (1 - I_{uu}), \quad G_1 = \eta_1 I_u,$$

so

$$p = G_1 \frac{1 - I_{uu}}{I_u} = G_1 \frac{2n + 1 - u_1^2}{2n + 1} \frac{n + 1}{u_1} . \qquad (3.43)$$

This p is inserted into (3.42) to give

$$\delta = \frac{2n + 1}{2n + 1 - u_1^2} \left(\frac{157}{G_1 \widetilde{Re}} \cdot \frac{1}{n + 1} \right)^{\frac{1}{n+1}} .$$

Then (3.41) readily gives

$$\zeta = \frac{8}{\widetilde{Re} G_1} \frac{2n + 1}{n + 1} \left(1 - \frac{u_1^2}{2n + 1}\right) \left[\frac{G_1 \widetilde{Re}}{157} (n + 1)\right]^{\frac{1-n}{1+n}} .$$

$$(3.44)$$

Figure 17 shows ζ as a function of M_* as calculated from (3.44).

There is a substantial rise in ζ after some decrease, which might at first sight appear to conflict with experiment [6, 31], which indicates that ζ decreases rapidly as the crisis is approached*. The conflict arises because the experiments were worked up in [6, 31] on the assumption that the flow was one-dimensional up to the crisis point. We shall see that this leads to results substantially different from those given by a more detailed study of the precrisis section.

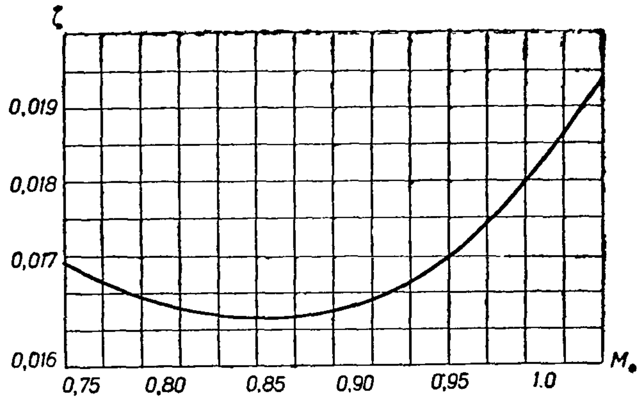

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Fig. 17.

Relation of ζ to M_*.

Let w be the mean flow speed; this is related to the pressure via (3.37) and (3.43) as follows:

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*Results analogous to those of [6, 31] were published elsewhere [59] as this manuscript was being prepared for press.

$$p = G_1 \frac{1 - w^2 + \frac{n^2}{2n + 1}}{w} . \qquad (3.45)$$

The above workers used this with n = 0, together with the equation of continuity, to deduce the mean speed from the measured pressure distribution. In fact, n is small and decreases as the crisis point is approached, so the speeds found in this way are very close to the true ones. On the other hand, the resulting ζ are substantially different from those given by the full theory.

The reason is essentially that within the framework of the one-dimensional theory it is impossible to obtain a speed in excess of that of sound.

We have already seen that the crisis occurs at speeds in excess of that of sound, although not by very much; but this radically alters all our concepts and causes a variation in ζ quite different from that derived from one-dimensional considerations.

This point is of considerable importance, so I consider it in somewhat more detail.

As in [6, 31], we deduce ζ from (2.25), but here we allow for possible change in the velocity profile.

From (2.25)

$$\zeta = \frac{8}{w\widetilde{Re}} \left(- \frac{k-1}{2kG_1} \frac{dp}{dx} - \frac{dw}{dx} \right) .$$

We derived dp/dx from (3.45) and convert from w to M_* on the basis of (1.47) to get

$$\zeta = \frac{4}{\widetilde{Re}} \frac{k+1}{k} (1 - M_*^2 + \xi) \frac{1}{M_*^3} \frac{dM_*}{dx} , \qquad (3.46)$$

in which

$$\xi = \frac{n^2}{2n+1}\left(1 - \frac{2}{n}\frac{n+1}{2n+1} M_* \frac{dn}{dM_*}\right). \qquad (3.47)$$

(3.46) shows that the change in profile causes us to add to the ζ found in the usual way (termed ζ_0)

$$\zeta_0 = \frac{4}{\tilde{R}e}\frac{k+1}{k}\frac{1-M_*^2}{M_*^3}\frac{dM_*}{dx} \qquad (3.48)$$

a correction $\Delta\zeta$:

$$\Delta\zeta = \frac{4}{\tilde{R}e}\frac{k+1}{k}\frac{\xi}{M_*^3}\frac{dM_*}{dx}, \qquad (3.49)$$

in which ξ is given by (3.47). We have seen in section 3 that $dn/du_1 \leq \leq 0$, so $dn/dM_* \leq 0$, and hence $\xi \geq 0$ at all times, whence from (3.49) we have $\Delta\zeta \geq 0$.

Then

$$\zeta = \zeta_0 + \Delta\zeta, \qquad (3.50)$$

in which the terms are given by (3.48) and (3.49).

$\Delta\zeta$ is not important while $M_* \ll 1$, the more so since $n \ll 1$; hence ξ is quite small.

However, $\zeta_0 \to 0$ when $M_* \to 1$, because (a difference from the one-dimensional case) $dM_*/dx \to \infty$ only when $M_* \to M_{*c} > 1$. Then the behavior of ζ near $M_* = 1$ is completely governed by that of $\Delta\zeta$, which (3.49) shows to increase as the crisis point is approached.

The treatment in [6, 31] gave ζ_0, which is close to ζ only if $M_* \ll 1$.

The ζ of (3.50) has, from (2.37), a definite physical significance: it is the ratio of the frictional stress at the wall to the mean momentum in that cross-section. This is merely a calculated quantity and has no direct physical existence. Deductions on the onset of crisis from boundary-layer theory require that ζ_0 should behave as in Fig. 18, as implied by (3.48). We must have $dM_*/dx \to +\infty$ for $M_* \to M_{*c} > 1$, so (3.48) gives $\zeta_0 \to -\infty$ for $M_* \to M_{*c}$. M_{*c} is close to one, so ζ_0 must fall rapidly as $M_* \to 1$, as is found [6, 31].

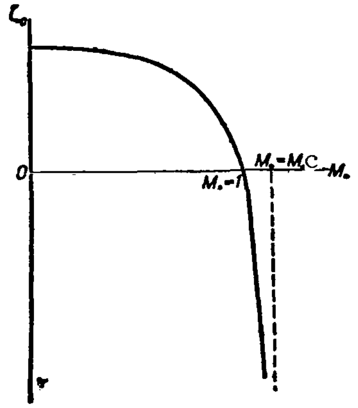


Fig. 18.

Relation of ζ_0 to M_*.

$\xi > 0$ always, so (3.49) gives $\Delta\zeta \to +\infty$ for $M_* \to M_{*c}$; then ζ tends to a definite limit as $M_* \to M_{*c}$ (Fig. 17), and the approach to the crisis point is accompanied by an increase in the resistance coefficient.

Note that (3.46) gives us a relation of n to u_1 near the crisis point. With $dM_*/dx = \infty$ we have

$$1 - M_*^2 + \xi = 0.$$

It can be shown that this equation agrees with (3.30).

These results show that the ζ appearing in the one-dimensional equations must be determined for the changing velocity profile near the crisis point. This means that we cannot determine ζ from the pressure change and conditions of continuity alone; experimental determination of ζ is meaningful only if the frictional stress is measured directly or if the velocity profile is recorded. In either case the experiments are very difficult, because even small perturbations can cause major disturbance near the crisis point.

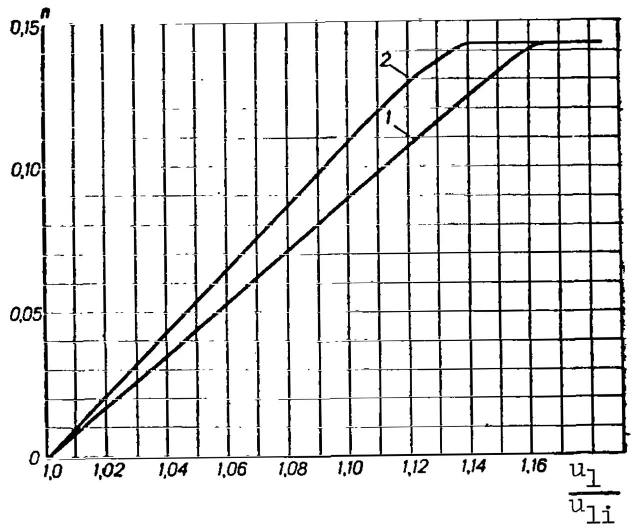
The experiments reported in [6] provide a general confirmation of the present theoretical deductions. These results [6] reveal an obvious discrepancy between the pressures measured on an end section of the tube and those calculated from the one-dimensional theory of the critical pressure. The actual pressure was less than the calculated critical value, which points to the attainment of supersonic velocities at the exit.

7. Initial Section of an Adiabatic Flow

Section 12 of chapter II implies that system (3.29) must be integrated subject to $f = 0$ for this case, the initial conditions being $n = 0$ and $u_1 = u_i$ at $x = 0$.

(3.35) describes the relation of n to u_1 in this section, and it indicates that $dn/du_1 > 0$ for $u_i^2 < [(k - 1)/(k + 1)]$ (subsonic speeds at inlet), so n increases with u_1 from 0 to 1/7, which corresponds to the initial (stabilization) section (for Re < 10^5). Figure 19 shows relations of n to u_1 calculated by numerical integration of (3.35) for u_i of 0.122 and 0.061, which correspond to M_{*i} of about 0.3 and 0.15.

This section then begins at u_1/u_{1i} of about 1.17 and 1.14 in the two cases.



[Commas represent decimal points]

Fig. 19.

Relation of n to u_1 for the initial section of an adiabatic flow; u_1:

1) 0.122 ($M_{*i} \approx 0.3$); 2) 0.061 ($M_{*i} \approx 0.15$)

The length of this section is found as follows: (3.29) gives

$$\frac{du_1}{dx} = \frac{\Delta_1}{\Delta},$$

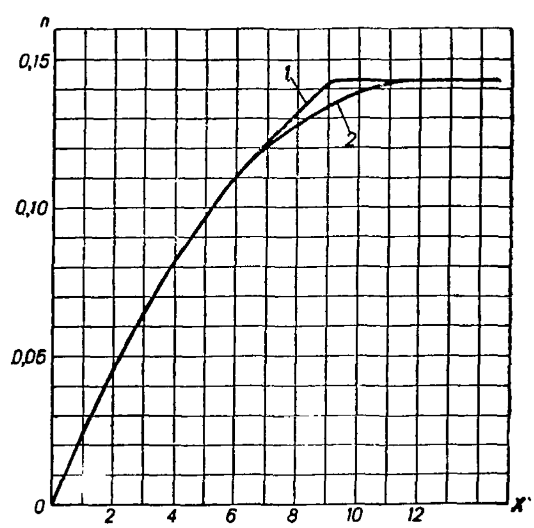
so

$$x = \int_{u_i}^{u_1} \frac{\Delta}{\Delta_1} du_1.$$

The relation of n to u_1 derived by integration of (3.35) should be used to compute this integral.

Figure 20 shows the relation of n to $\widetilde{Re}x/4$.

This gives a length much less than that found by experiment [23,49] (see chapter II), although the computations agree with theory [45]. Calculations on this part need further refinement.



[Commas represent decimal points]

Fig. 20.

Relation of n to ratio of length to hydraulic diameter of channel for M_{*i} of: 1) 0.3; 2) 0.15.

8. Crisis in a Flow with Heat Transfer

Here we derive relationships analogous to those of section 3, but with $m \neq 0$ and $\Theta \neq 1$ (heat transfer present).

The determinant of (3.22) must be equated to zero to find the conditions of onset. From Table 1 for $Z_{13} = Z_{14} = Z_{31} = Z_{42} = Z_{44} = 0$, we obtain

$$\begin{vmatrix} Z_{11} & Z_{12} & 0 \\ Z_{21} & Z_{22} & Z_{24} \\ 0 & Z_{32} & Z_{34} \end{vmatrix} = 0. \tag{3.51}$$

Inserting the coefficients and rejecting the trivial solution $\Theta_1 = \Theta_0^*$, we have

$$\frac{2k}{k-1} u_1^2 (I - u_1^2) - (I + u_1^2)(\Theta_1 - u_1^2) -$$

$$- (I - \Theta_1)\left[(2n+1) I + u_1^2 + m\left(\frac{2n+1}{m+1}\right)^2 (\Theta_0 - \Theta_1) \right] = 0,$$

or

$$u_1^4 - 2B' u_1^2 + C' = 0, \tag{3.52}$$

*System (3.22) consists of two equations, not four, if $\Theta_1 = \Theta_0$.

in which

$$B' = \frac{k}{k+1} I,$$

$$C' = \frac{k-1}{k+1} \left[I + (2n+1) I (I-1) + m \left(\frac{2n+1}{m+1}\right)^2 (\Theta_0 - \Theta_1)(I - \Theta_1) \right]$$

as I is given by (3.23).

As we would expect, (3.52) becomes the particular case (3.30) when $\Theta_1 = \Theta_0$ (or $m = 0$).

Now u_1 is referred to the limiting speed at the center of the channel at the inlet. The stagnation temperature Θ of the gas, and hence also this speed, varies along the length and along the width, so it is best to use the speed referred to the corresponding limiting speed in order to characterize effects in any section of the channel, including the critical one. From (1.46) we put

$$V = \frac{u}{u_1} = \frac{u}{\sqrt{\Theta}} .$$

But $V \leq V_1$ if the gas is heated; in fact, in this case $\Theta_1 \leq \Theta \leq \Theta_0$, so

$$V \leq \frac{u}{\sqrt{\Theta}} \leq \frac{u_1}{\sqrt{\Theta}} = V_1 .$$

We divide (3.52) by Θ_1^2 and put

$$J = \frac{I}{\Theta_1} = \frac{2n+1}{m+1} \left(m \frac{\Theta_0}{\Theta_1} + 1 \right),$$

to get

$$V_1^4 - 2BV_1^2 + C = 0, \tag{3.53}$$

in which

$$B = \frac{k}{k + 1} J,$$

$$C = \frac{k - 1}{k + 1} \left[J + (2n + 1) J (J - 1) + m \left(\frac{2n + 1}{m + 1} \right)^2 \left(\frac{\Theta_0}{\Theta_1} - 1 \right) (J - 1) \right.$$

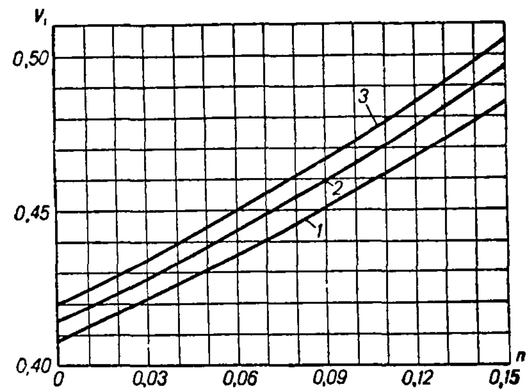
As in section 3, we can show that the required solution of (3.53) is

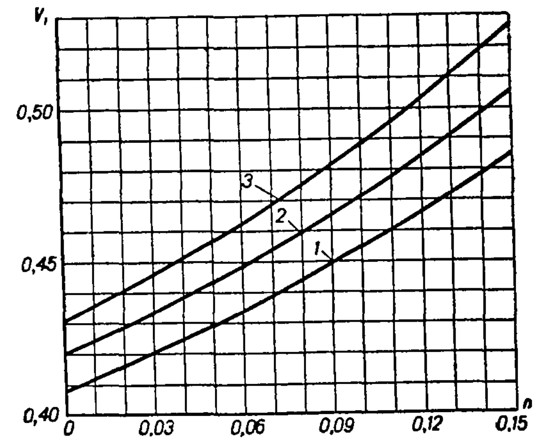
$$V_1^2 = B - \sqrt{B^2 - C}. \tag{3.54}$$

There is a definite relation of u_1 to n for adiabatic flow; (3.53) shows that the relation of u_1 to m and n at the crisis point is also affected by Θ_0/Θ_1 in the case of heat transfer.

Figures 21 and 22 show results from calculations for V_1 at the crisis point for Θ_0/Θ_1 of 1.5 and 2. The crisis point can occur for various combinations of m and n; as for adiabatic flow, we have to consider the flow in the whole channel to determine whether it will actually occur.

I will show that the two profiles flatten out as the crisis point is approached if the gas does not cool. From (3.22) we find that





[Commas represent decimal points]

Fig. 21.

Relation of V_1 (axial velocity at crisis point) to m and n for $\Theta_0/\Theta_1 = 1.5$ and m of: 1) 0; 2) 0.06; 3) 0.12.

Fig. 22.

Relation of V_1 to m and n for $\Theta_0/\Theta_1 = 2$ for m of: 1) 0; 2) 0.06; 3) 0.12.

$$\frac{dn}{du_1} = \frac{dn}{dx} : \frac{du_1}{dx} = \frac{\Delta_2}{\Delta_1},$$

$$\frac{dm}{du_1} = \frac{dm}{dx} : \frac{du_1}{dx} = \frac{\Delta_3}{\Delta_1},$$

in which the Δ are determinants. We substitute for the coefficients and apply (3.53) for the crisis point to get that

$$\frac{dn}{du_1} = - \frac{n+1}{u_1} \frac{2n+1}{\Theta_1 - u_1^2} \Theta_1 (J - 1), \tag{3.55}$$

$$\frac{dm}{du_1} = \frac{m}{n+1} \frac{dn}{du_1} . \tag{3.56}$$

$\Theta_0 \geq \Theta_1$ if the gas does not cool, and

$$J = \frac{2n+1}{m+1} \left(m \frac{\Theta_0}{\Theta_1} + 1 \right) \geq 2n + 1 > 1,$$

so (3.55) gives $dn/du_1 < 0$, and (3.56) gives $dm/du_1 < 0$. Continuity implies that these inequalities must also apply in the neighborhood of the crisis point, so the two profiles flatten out as that point is approached.

9. Transition Through the Speed of Sound Near the Axis of the Channel

Figures 21 and 22 show that the speed at the axis exceeds the speed of sound at the point of crisis when $m \neq 0$ and $n \neq 0$. It is also easy to show that the u_m of (3.37) exceeds that speed at that point.

This possibility is related to the special behavior of the stream lines in the precrisis region, as in the case of adiabatic flow.

This can be demonstrated from the theory of the boundary layer [26].

For simplicity I consider only laminar flow, because turbulent flow does not differ essentially from laminar, and we are interested only in qualitative results.

From (2.11) and (2.12) we have

$$p = \rho\,(\Theta - u^2),$$

so

$$\frac{dp}{dx} = (\Theta - u^2)\,\frac{\partial \rho}{\partial x} + \rho\,\frac{\partial \Theta}{\partial x} - 2u\rho\,\frac{\partial u}{\partial x}\,. \qquad (3.57)$$

From (2.10), the equation of continuity, we have

$$\frac{\partial \rho}{\partial x} = -\,\frac{\rho}{u}\left(\frac{\partial u}{\partial x} + \tilde{Re}\,\frac{\partial v}{\partial y}\right) - \tilde{Re}\,\frac{v}{u}\,\frac{\partial \rho}{\partial y}\,,$$

in which from (2.11) and (2.12) we can make the substitution

$$\frac{\partial \rho}{\partial y} = \frac{p}{(\Theta - u^2)^2}\left(2u\frac{\partial u}{\partial y} - \frac{\partial \Theta}{\partial y}\right) = \frac{\rho}{\Theta - u^2}\left(2u\frac{\partial u}{\partial y} - \frac{\partial \Theta}{\partial y}\right),$$

so

$$\frac{\partial \rho}{\partial x} = \tilde{Re}\,\frac{v\rho}{u(\Theta - u^2)}\left(\frac{\partial \Theta}{\partial y} - 2u\frac{\partial u}{\partial y}\right) - \frac{\rho}{u}\left(\frac{\partial u}{\partial x} + \tilde{Re}\,\frac{\partial v}{\partial y}\right).$$

This $\partial\rho/\partial x$ is inserted in (3.57) and (2.12) is used to give

$$\frac{dp}{dx} = \tilde{Re}\,\frac{v\rho}{u}\left(\frac{\partial \Theta}{\partial y} - 2u\frac{\partial u}{\partial y}\right) - \frac{\rho T}{u}\left(\frac{\partial u}{\partial x} + \tilde{Re}\,\frac{\partial v}{\partial y}\right) + \rho\frac{\partial \Theta}{\partial x} - 2u\rho\frac{\partial u}{\partial x}.$$

Then (2.12) is replaced by

$$u\frac{\partial u}{\partial x} + \tilde{Re}\,v\frac{\partial u}{\partial y} = \frac{k-1}{2k}\frac{T}{u}\left(\frac{\partial u}{\partial x} + \tilde{Re}\,\frac{\partial v}{\partial y}\right) + \frac{k-1}{2k}\left(2u\frac{\partial u}{\partial x} - \frac{\partial \Theta}{\partial x}\right) +$$

$$+ \frac{k-1}{2k}\tilde{Re}\,\frac{v}{u}\left(2u\frac{\partial u}{\partial y} - \frac{\partial \Theta}{\partial y}\right) + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right),$$

or

$$u\frac{\partial u}{\partial x} - \frac{k-1}{2k}\frac{T}{u}\frac{\partial u}{\partial x} - \frac{k-1}{k}u\frac{\partial u}{\partial x} = -\tilde{Re}\,v\frac{\partial u}{\partial y} + \frac{k-1}{2k}\frac{T}{u}\tilde{Re}\,\frac{\partial v}{\partial y} -$$

$$-\frac{k-1}{2k}\left(\frac{\partial \Theta}{\partial x} + \tilde{Re}\,\frac{v}{k}\frac{\partial \Theta}{\partial y}\right) + \frac{k-1}{k}\tilde{Re}\,v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right).$$

This last, transformed slightly with the aid of (2.9), may be put as

$$\frac{u^2 - \frac{k-1}{2}T}{ku}\frac{\partial u}{\partial x} = -\tilde{Re}\frac{v}{k}\frac{\partial u}{\partial y} + \frac{k-1}{2k}\tilde{Re}\frac{T}{u}\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) -$$

$$-\frac{k-1}{2k}\left(\frac{\partial\Theta}{\partial x} + \frac{v}{k}\frac{\partial\Theta}{\partial y}\right) + \frac{k-1}{k}\tilde{Re}\, v\frac{\partial u}{\partial y},$$

or, with $M^2 = 2u^2/T(k - 1)$ and further manipulation,

$$\frac{k-1}{2k}\frac{T}{u}(1 - M^2)\frac{\partial u}{\partial x} = \tilde{Re}\frac{v}{k}\frac{\partial u}{\partial y} - \frac{\mu}{k\rho}\frac{\partial^2 u}{\partial y^2} + \frac{k-1}{2k}\frac{\mu}{\rho u}\frac{\partial^2\Theta}{\partial y^2} -$$

$$-\frac{1}{\rho}\frac{\partial\mu}{\partial y}\frac{\partial u}{\partial y} + \frac{k-1}{2k}\frac{1}{\rho u}\frac{\partial\mu}{\partial y}\frac{\partial\Theta}{\partial y} - \frac{k-1}{2k}\tilde{Re}\frac{T}{u}\frac{\partial v}{\partial y}. \qquad (3.58)$$

Here $\partial u/\partial y > 0$, because the speed increases from wall to axis, and $\partial^2 u/\partial y^2 < 0$, since the velocity profile is convex downstream. In the case of heating, $\partial\Theta/\partial y < 0$ and $\partial^2\Theta/\partial y^2 > 0$.

Further,

$$\frac{\partial\mu}{\partial y} = \frac{\partial T}{\partial y}\frac{d\mu}{\partial y} = \left(\frac{\partial\Theta}{\partial y} - 2u\frac{\partial u}{\partial y}\right)\frac{d\mu}{dT} < 0,$$

because

$$\frac{d\mu}{dT} > 0, \quad \frac{\partial \Theta}{\partial y} < 0, \quad \frac{\partial u}{\partial y} > 0.$$

$v > 0$ at the start; $v = 0$ at the axis, so $\partial v/\partial y > 0$ near the axis.

Then all the terms on the right in (3.58) are positive near the axis, so $\partial u/\partial x > 0$ (velocity near the axis increases) for $M < 1$.

Balancing of the heating and frictional effects is possible only if $v < 0$ for $M = 1$ and later points. This means that a stream line that approaches the axis at the start will come closest to it at some point and then will bend away again.

CHAPTER IV

RESULTS OF COMPUTATIONS

1. Selection of Conditions and Computation Parameters

The general deductions made in chapter III lead us to perform computations designed to elucidate the effects of various types of heat influx on the coefficients of heat transfer and resistance, and also on the onset of the crisis. I begin with $m = n = 0$ (initial part).

We put $\beta = 1$ in (2.35) for simplicity; (2.79) is solved (the wall temperature is assumed given). $Re > 10^5$ is avoided, and also the region of very low Re, by means of a preliminary estimate of the variation in Re along the channel; the one-dimensional theory can be used for this.

From (2.57) we have

$$Re = 4\,\widetilde{Re}\,\frac{w\rho}{\mu},$$

so, using the equation of continuity with $\tau = T/\Theta$, we have

$$\frac{Re}{Re_i} = \frac{w\rho}{w_i\rho_i}\,\frac{\mu_i}{\mu} = \frac{\mu_i}{\mu} = \frac{T_i}{T} = \frac{\tau_i}{\tau}\,\frac{1}{\Theta},$$

because the temperature scale has been chosen to make $\Theta_i = 1$. Re_f/Re_i will be governed by M_{*f} and Θ_0 if the flow is taken as far as the crisis point (i.e., to $M_{*f} = 1$):

$$\frac{Re_f}{Re_i} = \frac{\tau_i}{\tau(1)} \frac{1}{\Theta_f} . \qquad (4.1)$$

For adiabatic flow

$$\left(\frac{Re_f}{Re_i}\right)_a = \frac{\tau_i}{\tau(1)} .$$

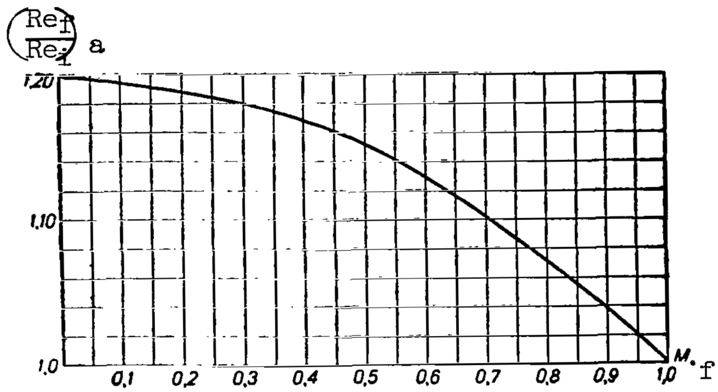


Fig. 23.

Relation of ratio of Re_f to Re_i against M_{*i} for adiabatic flow; $M_{*f} = 1$.

Figure 23 implies that $(Re_f/Re_i)_a$ is sufficiently close to one and varies little with M_{*i} for $M_{*i} < 0.5$, so we are reasonably free to choose M_{*i}. Reasonable limiting lengths for the channels are obtained, together with only minor effects from compressibility at the inlet, by putting $M_{*i} = 0.3$; then $Re_f/Re_i \approx 1.18$.

(4.1) shows that $Re_f < Re_i$ for sufficiently high degrees of heating; but $Re_i < Re_f$ for adiabatic flow, so Re_i must be chosen from the condition that $Re_f^a < 10^5$, so $Re_f < Re_i < Re_f^a < 10^5$. With $Re_i = 8 \times 10^4$ we have $Re_f^a < 1.18 \times 8 \times 10^4 < 10^5$.

(1.63) gives us Θ_f, for $\Theta_f < \Theta_f|_{\xi=0} \approx 6.8$, so (4.1) gives $Re_f > 1.4 \times 10^4$.

I consider only cases in which the temperature does not decrease along the length.

With $\Theta_{oi} = 1.1$ and $\Theta_{of}/\Theta_{oi} < 5$ we get $\Theta_f < \Theta_{of} < 5\Theta_{oi} = 5.5$.

Then a rise in wall temperature by not more than a factor five is bound to give us the required range of variation in gas temperature.

The limiting length of the channel can be estimated with $\zeta = 0.022$, which corresponds to an average Re of about 4×10^4.

Heat supply at the exit gives us $\widetilde{Re}\bar{L}_1/4 = 265$, from (1.55), (1.56), and (2.7) together with the fact that the hydraulic diameter is twice the width of the channel.

The limiting length is less in the general case of heat transfer, so, if we assume that the wall temperature reaches $\Theta_{of} = 5\Theta_{oi}$ when $\widetilde{Re}x/4 = 265$, we certainly have $\Theta_{of} < 5\Theta_{oi}$ when x reaches its limiting value.

Linear and exponential forms are taken for the relation of Θ_0 to x:

$$\Theta_0 = \Theta_{oi} + A_1 \frac{1}{4} \widetilde{Re}x,$$

$$\Theta_0 = \frac{1}{2}\,\Theta_i\left(1 + e^{A_2 \frac{1}{4}\widetilde{\text{Re}}x}\right).$$

With $\Theta_{oi} = 1.1$ and $\Theta_0 = 5\Theta_{oi}$ for $\widetilde{\text{Re}}x/4 = 265$, we have two equations for Θ_0:

$$\Theta_0 = 1.1 + 0.015\,\frac{1}{4}\,\widetilde{\text{Re}}x, \qquad (4.2)$$

$$\Theta_0 = \frac{1.1}{2}\left(1 + e^{0.0083\,\frac{1}{4}\,\widetilde{\text{Re}}x}\right). \qquad (4.3)$$

It is also of interest to examine a fixed small wall temperature $\Theta_o = 1.1$ and also adiabatic flow ($\Theta_0 = 1$).

There are then four distinct distributions:

I. $\Theta_0 = 1,$

II. $\Theta_o = 1.1,$

III. $\Theta_0 = 1.1 + 0.015\,\frac{1}{4}\,\widetilde{\text{Re}}x,$

IV. $\Theta_0 = \frac{1.1}{2}\left(1 + e^{0.0083\,\frac{1}{4}\,\widetilde{\text{Re}}x}\right).$

It remains to find the initial conditions and the constants.

We assume $m = n = 0$ for $x = 0$, also, $\Theta_{1i} = 1$, in accordance with the choice of temperature scale, and $M_{*i} = 0.3$, which corresponds to

$u_i = 0.1224.$

The initial conditions identical for all four cases for $x = 0$ are then

$$u_i = 0.1224, \quad \Theta_1 = 1, \quad m = n = 0.$$

$\widetilde{Re}$ and G_1 are also the same for all.

We have from (1.57), $\rho = p/T$, and $\mu = T$ that

$$\widetilde{Re} = \frac{1}{4} Re_i \frac{\mu_i}{u_i \rho_i} = \frac{1}{4} Re_i \frac{T_i^2}{u_i p_i} .$$

Here

$$p_i = \Pi(0.3) = 0.9485, \quad T_i = \tau(0.3) = 0.9850, \quad u_i = 0.1224$$

and so $\widetilde{Re} = 1.68 \times 10^5$.

G_1 is found because $m = n = 0$ at the inlet, so

$$G_1 = \frac{\Pi(M_{*i})u_i}{1 - u_i^2} = \sqrt{\frac{k - 1}{k + 1}} \left(\frac{2}{k + 1}\right)^{\frac{1}{k-1}} g(M_{*i}).$$

For air

$$G_1 = 0.259 g(M_{*i}),$$

so

$$G_1 = 0.259\ g\ (0.3) \approx 0.118.$$

2. Sequence of Computation

A numerical method may be used to program (2.79) for computer solution. The following feature must be taken into consideration here.

The computation for the initial and main sections is performed in three stages.

Stage I

We assume $f = F = 0$, this stage ends when we obtain either $n = 1/7$ or $m = 1/7$.

Stage II

1. If $n = 1/7$ at the end of stage I, we have $n = 1/7$ and $F = 0$ everywhere in stage II, while f is given by the condition of compatibility for (2.79):

$$f = \begin{vmatrix} N_{11} & N_{14} & N_1'' \\ N_{21} & N_{24} & N_2'' \\ N_{31} & N_{34} & N_3'' \end{vmatrix} \begin{vmatrix} N_{11} & N_{14} & N_1' \\ N_{21} & N_{24} & N_1' \\ N_{31} & N_{34} & N_3' \end{vmatrix}$$

The integration is continued until $m = 1/7$.

2. If $m = 1/7$ at the end of stage I, we have $m = 1/7$ and $f = 0$ everywhere in stage II, while $F = (d\Theta_1/dx)/N'_4$.

The completion of stage II brings us to the end of the initial section.

Section III

This stage corresponds to the main section, which is computed with $m = n = 1/7$, f and F being found from the condition of compatibility in accordance with section 10 of chapter II.

3. Adiabatic Flow

Figures 24-28 give results for adiabatic flow (case I of section 1)*.

Figures 24 and 25 show that $f = 0$ when $u_1 = 0.4$, so n cannot remain constant for $u_1 > 0.4$ if we assume that (2.79) still applies, since this would imply negative f, which is impossible [see (2.82)]. This means that we must assume that n varies for $u_1 > 0.4$. The only acceptable assumption for f in boundary-layer theory is $f = 0$ for $u_1 > 0.4$.

Then df/dx (for f as a function of X) is discontinuous at $X \cong 345.5$ (Fig. 26; this corresponds to $u_1 \cong 0.4$), so it would be more correct to assume that n starts to vary as $f \to 0$.

However, this introduces no major change into the results: Fig. 27 /10 shows that n varies only very slowly for $X > 345.5$, which indicates that flattening of the profile in the precrisis section. The qualitative deductions of chapter III are thus confirmed. Figure 28 shows the change in the velocity profile in the precrisis part.

Special attention must be given to the results for the initial

*All curves in this chapter are given as functions of $X = \widetilde{\mathrm{Re}}x/4$ (i.e., of the current length of the channel referred to the hydraulic diameter).

section, whose length is found to be small relative to values indicated by experiment [23, 49], as in chapter III.

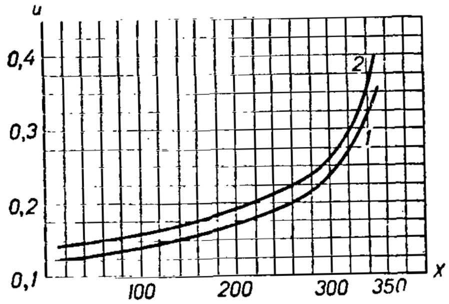


Fig. 24.

Mean speed u_m and speed at channel axis u_1 as functions of X with adiabatic flow for the main section: 1) u_m; 2) u_1.

However, the results obtained here show that better agreement with experiment is obtained if the computation is performed with some care. Figure 25 shows that f jumps from 0 to 0.963 at X = 12.6 (junction of initial and main parts); if we eliminate this discontinuity (e.g., by a method analogous to that given in [29] for laminar flow), we find that the length of the initial section increases.

4. Flow With Heat Transfer

Cases II-IV of section 1 apply here; Fig. 29 shows that $m > n$ in the initial section in cases II and IV, while $m < n$ there for case III. In case IV m and n become 1/7 almost simultaneously; in II, m reaches 1/7 earlier than n; and in III, n reaches 1/7 before m.

The various modes of inflow of heat thus cause different modes of variation in the form parameters in the initial section.

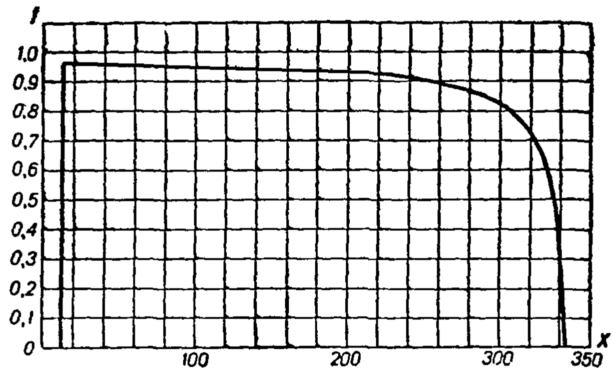


Fig. 25.

Relation of f to current length of channel for adiabatic flow.

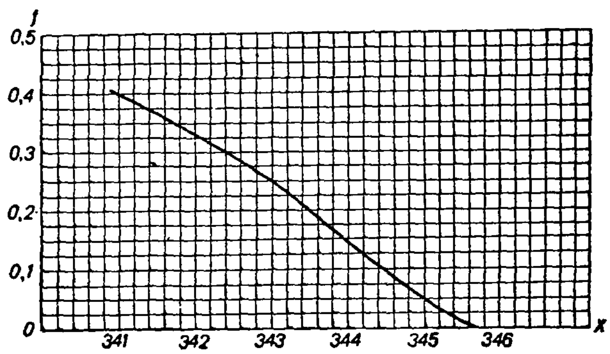


Fig. 26.

Relation of f to current length of channel for adiabatic flow in the precrisis section.

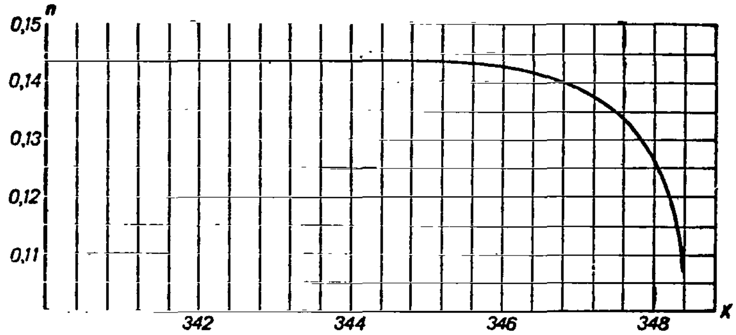


Fig. 27.

Variation of n in and near the precrisis section for adiabatic flow.

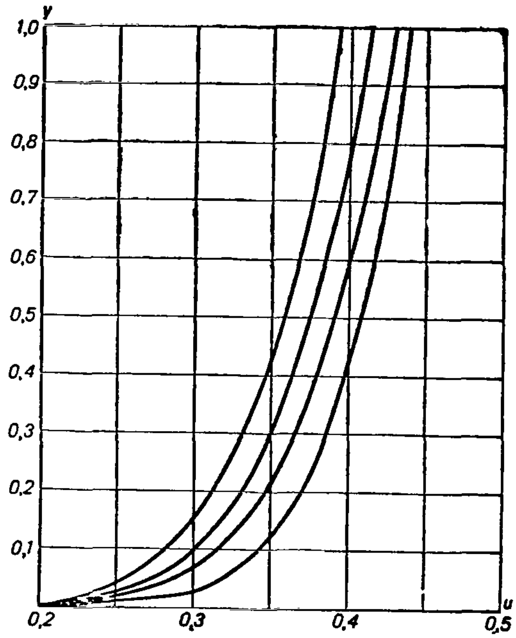


Fig. 28.

Velocity profiles in the precrisis section.

F and f are both different from zero in the main section; there is a stepwise change in f or F at the transition from the initial section (Fig. 30).

The discontinuity in f or F leads (as in case I) to a considerable reduction in the length of the initial section as compared with

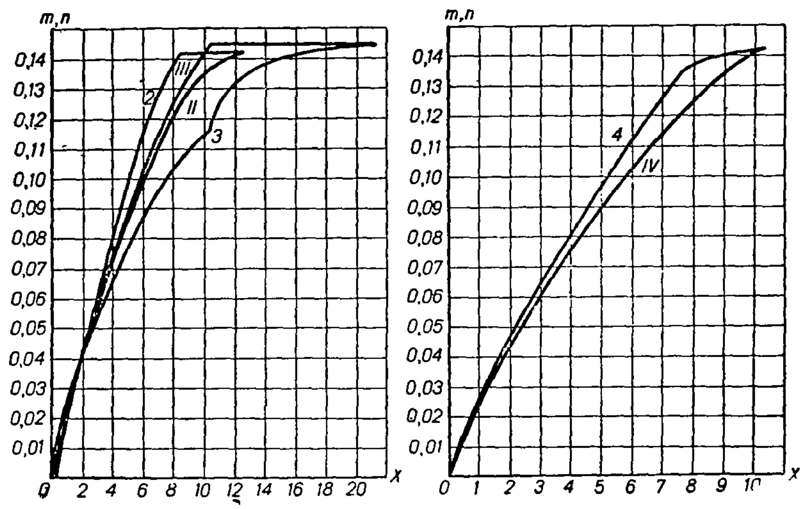


Fig. 29.

Relation of m and n to X for the initial section of channel: 2), 3), and 4) are for m; II), III), and IV) are for n. 2) and II) are for case II; 3) and III) are for case III; and 4) and IV) are for case IV.

experiment.

The calculations on the initial section need further refinement, but the results already show that substantially different types of flow can occur there in response to different modes of influx of heat.

The derivatives of the form parameters are discontinuous, as are f and F, but the transition to the main section is reasonably smooth in all cases (Fig. 30), which indicates that a revised calculation for the initial section would (at least in some cases) not lead to any substantial change in the velocity and temperature as functions of X.

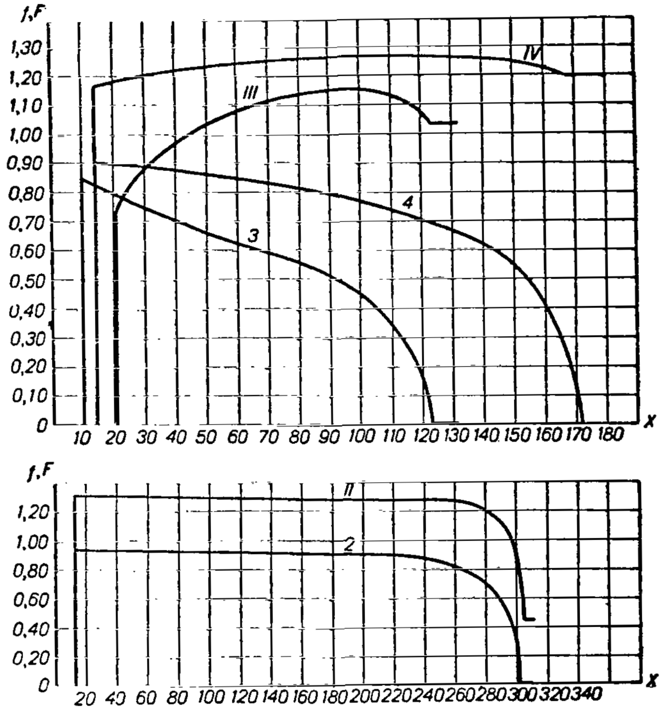


Fig. 30.

Relation of f and F to X:2), 3), and 4) are for f; II), III), and IV) are for F. 2) and II) are for case II; 3) and III) are for case III; and 4) and IV) are for case IV.

At some $X = X_f$ (which varies from case to case) we find that f

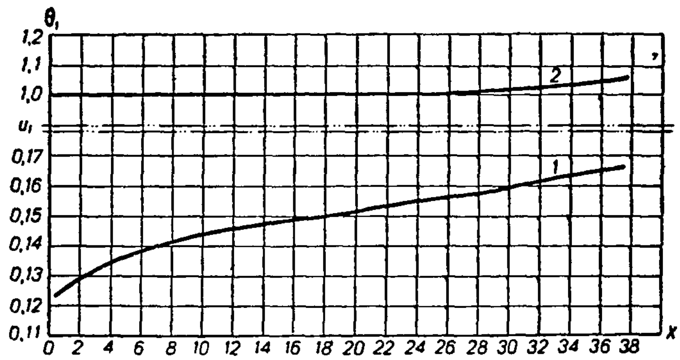


Fig. 31.

Relation of u_1 and Θ_1 at channel axis to X near the junction between the initial and main sections for case III: 1) u_1; 2) Θ_1.

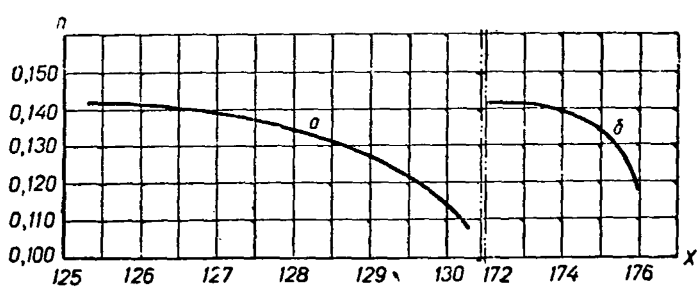


Fig. 32.

Relation of n to X for the pre-crisis section:

a) case III; b) case IV.

becomes zero for all cases with heat transfer; calculations with $m = n = 1/7$ would then make f negative, so flow in the range $X > X_f$ with $m = n = 1/7$ would be impossible. We must assume that $f = 0$ for $X > X_f$. A difference from the adiabatic case is that various assumptions can

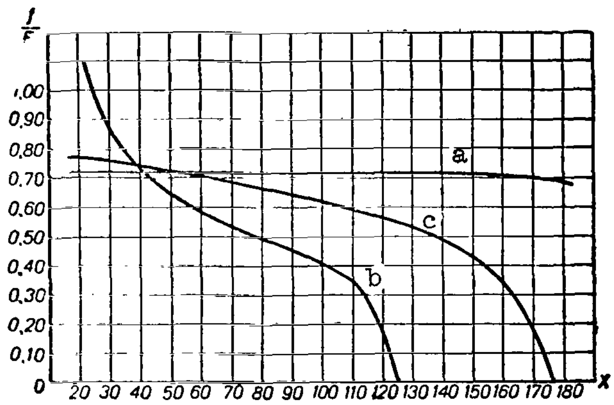


Fig. 33.

Relation of f/F to X: a) case II; b) case III; c) case IV.

be made about the behavior of m and n for $X > X_f$. It can be shown that we cannot have $m = n$ at the crisis point and that m varies much more slowly than n near that point. On this basis the precrisis section has been computed subject to the simplifying assumption that n varies, $f = 0$, and m remains constant. Figure 32 shows the relation of n to X for the precrisis section, from which we see that n in these cases varies much as in adiabatic flow.

The results show that Reynolds' analogy is almost correct over

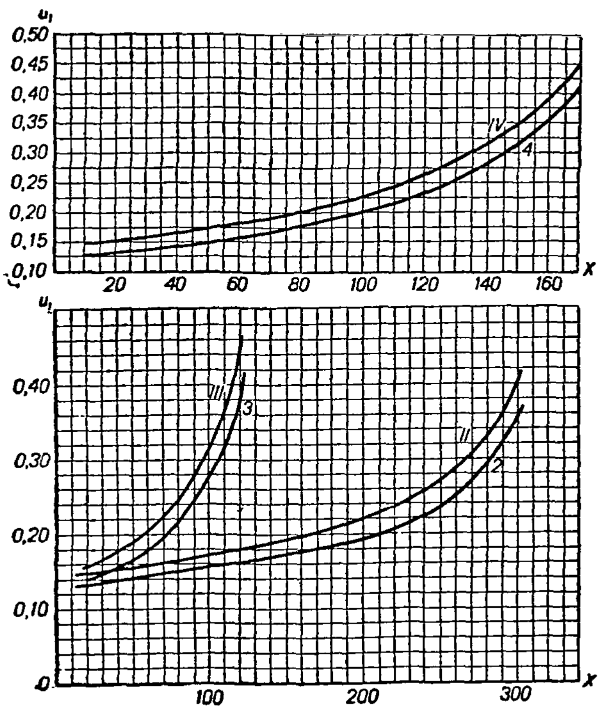


Fig. 34.

Relation of u_m and u_l to X for the main section: 2), 3), and 4) are for u_m; II), III), and IV) are for u_l. 2) and II) are for case II: 3) and III) are for case III; and 4) and IV) are for case IV.

a fairly wide range (in the main section of the flow); but the method does not allow us to establish precisely where this analogy begins to fail.

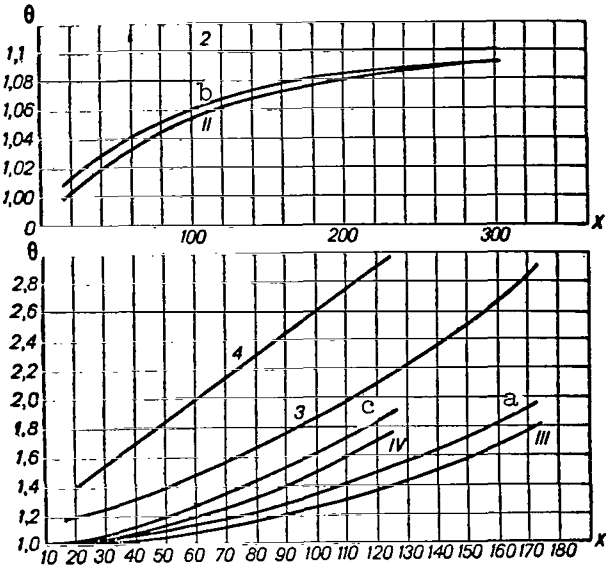


Fig. 35.

Relation of wall temperature Θ_0, stagnation temperature at axis Θ_1, and mean stagnation temperature Θ_m to X for the main section:

2)-4) Θ_0; II)-IV) Θ_1; b)-d) Θ_m; 2), II), and b) case II; 3), III), and c) case III; 4), IV), and d) case IV.

The following factors may lead to fairly early failure of the analogy. Figure 33 shows f/F as a function of X for cases II, III, and IV; f/F represents the ratio of the curvatures of the velocity and temperature profiles at the axis, and it is clear that this ratio decreases continuously and is always less than one, so the two profiles at the axis are not similar in the strict sense even at the very start of the main section.

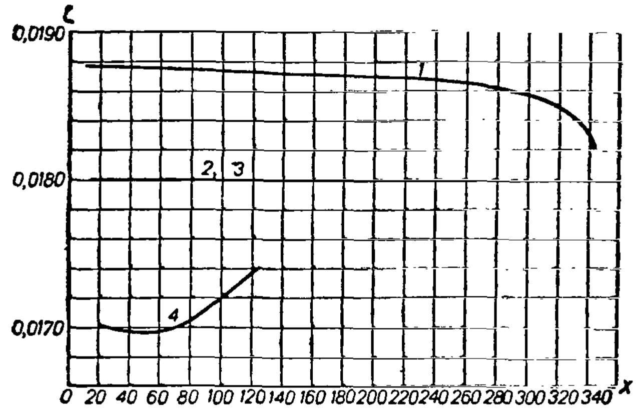


Fig. 36.

Relation of ζ to X for the main part: 1) case I; 2) case II; 3) case III; 4) case IV.

The discrepancy increases with the speed and ultimately causes the two profiles to lack any similarity at all.

Figure 34 shows the velocity at the axis and the mean velocity as functions of X for the main part for cases II, III, and IV.

Figure 35 shows the relation of wall temperature, stagnation temperature at axis, and mean stagnation temperature to X for the main part for the same cases.

5. Coefficients of Resistance and Heat Transfer

The computations for the initial and precrisis sections need refinement, so I give results only for the main section.

Figure 36 shows that ζ in all cases does not vary greatly within the main section, but the mode of variation differs from one case to another. The differences cannot be explained solely in terms of differences in Re and M; the different modes of influx of heat play a major part.

Reynolds' analogy applies to the main section (our basic assumption), so the heat-transfer coefficient varies there in the same manner as the resistance coefficient.

CONCLUSIONS

Calculations on turbulent gas flows in channels in the presence of heat transfer are very complicated; great difficulties arise in obtaining practical solutions even in the simplest cases, because the theory of turbulence is far from being completely worked out. Practical requirements demand suitable approximate methods.

One of these is the semiempirical method long used in hydrodynamics; its basis is that experimental evidence obtained over a restricted range of conditions can throw light on the effects of those conditions over a wider range.

Here I have tried to develop a semiempirical method of calculating subsonic flows in pipes subject to an arbitrary influx of heat through the wall.

The following experimental evidence is used:

1. The form parameter for the velocity profile has a clear-cut relation to Re for fairly low subsonic speeds in the absence of heat transfer; in particular, this parameter is constant for Re $< 10^5$.

2. Reynolds' analogy applies for a certain range of variation in the conditions at the wall.

3. The thickness of the viscous sublayer is given by $u_\delta \delta/\nu = Re_\delta$ in the absence of heat transfer; here Re_δ is the turbulence constant. It is assumed that Re_δ remains unchanged in the presence of heat transfer, provided that some mean value is taken for the viscosity of the sublayer.

This method may be extended: we may replace Reynolds' analogy by the assumption that there is a definite relation between the parameters for the velocity and temperature profiles.

Some of the assumptions made in the deduction of the equations are not essential; for instance, the relation of viscosity to temperature may be taken as $\mu = CT^\beta$ or $\mu = f(T)$ instead of $\mu = T^\beta$. In addition, it would be possible to take account of the deviation from the turbulent profile in the sublayer in deducing the mean speed and temperature.

In every case the problem amounts to integration of a system with unknown velocity and temperature at the axis and with two form parameters. The equations are quasilinear in the derivatives.

Calculations for Re $< 10^5$ have shown that the flow in a channel can be divided into three parts: initial (curvature of profiles at the axis zero), main (form factors are the same and constant), and precrisis (form factors cannot be constant).

The most reliable results are obtained for the main part. More precise calculations are needed for the other parts.

The method is based on the assumption that the boundary-layer equations apply everywhere in the channel.

There is no especial doubt over this for the initial part (except for X very small), so the calculation for this part may be refined within the framework of boundary-layer theory; but there are great difficulties over the precrisis part.

The longitudinal velocity increases very rapidly near the crisis point (except in an area near the wall), so the corresponding stresses must be incorporated in the equations of motion, although it is difficult to say much about these for the case of turbulent flow at present. The very rapid fall in pressure in this part requires a correction for the effect of the pressure gradient on the velocity profile in the sublayer*. In addition, it is hardly possible to consider the pressure independent of the transverse coordinate near the crisis point [6].

This all indicates that we should accept only with reserve the conclusions derived by the usual methods of boundary-layer theory for the precrisis part.

The length of this part is not very large relative to that of the main part, so a refined calculation for it is hardly to be reckoned of great practical value.

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*Calculations show that dp/dx has little effect on the velocity profile in the sublayer in the main part.

APPENDIX I

JUSTIFICATION FOR THE OPERATIONS ON SERIES IN CHAPTER II

The term-by-term integration and differentiation in chapter II can be justified if we can show that the series after integration or differentiation in this way converge uniformly.

Proof of uniform convergence is given by use of Weierstrass's test.

The series

$$1 + \frac{u^2}{\Theta} + \frac{u^4}{\Theta^2} + \ldots = 1 + V^2 + V^4 + \ldots$$

converges uniformly, because $V \leq V_1 \leq V_{1cr} < 1$, so the sums of the series of (2.48), (2.49), and (2.50) are respectively I_1, I_2, and I_3.

The series for $u^i\Theta^j$ have the general term

$$a_k(y) = u_1^i \Theta_0^f \binom{j}{k} \left(\frac{\Theta_1}{\Theta_0} - 1 \right)^k y^{mk+ni} .$$

For any $j < 0$

$$\left| \binom{j}{k} \right| = \frac{|j|(|j| + 1)(|j| + 2)\ldots(|j| + k - 1)}{k!} <$$

$$< (k + 1)(k + 2)\ldots(k + |j|-1) \leq (k + |j| - 1)^{|j|-1} ,$$

so for $0 \le y \le 1$ and $\Theta_1/\Theta_0 < 1$ we have

$$|a_k(y)| < u_1^i \Theta_0^j (k + |j| - 1)^{|j|-1} \left(1 - \frac{\Theta_1}{\Theta_0}\right)^k .$$

Now for $\Theta_1/\Theta_0 < 1$

$$\lim_{k \to \infty} (k + |j| - 1)^{|j|-1} \left(1 - \frac{\Theta_1}{\Theta_0}\right)^{\frac{k}{2}} = 0,$$

so we can find an N such that for all $k \ge N$ we have, by virtue of the bounds to u_1 and Θ_0, that

$$u_1^i \Theta_0^j (k + |j| - 1)^{|j|-1} \left(1 - \frac{\Theta_1}{\Theta_0}\right)^{\frac{k}{2}} < 1,$$

and so for $k \ge N$ that

$$|a_k(y)| < \left(\sqrt{1 - \frac{\Theta_1}{\Theta_0}}\right)^k ,$$

which demonstrates the uniform convergence of the series for $u^i \Theta^j$ for $0 \le y \le 1$.

Consider the series obtained in section 10 after differentiating I_1, I_2, and I_3 with respect to z_1, z_2, z_3, z_4, and Θ_0; here $0 \le y \le 1$ and $\Theta \ge \Theta_1$, so

$$I^{(\iota,j)} = \int_0^1 u^i \Theta^j dy = u_1^i \int_0^1 \frac{y^{ni}}{\Theta \lceil j \rceil} dy < \frac{u_1^i}{\Theta_1 \lceil j \rceil} = \frac{z_1^i}{\Theta_1 \lceil j \rceil} ,$$

and so for $k > 1$ the general terms of the series for $\partial I_\iota / \partial z_1$ ($\iota = 1,2,3$) will be

$$\frac{k}{z_1} I^{(k,j)} < \frac{k}{z_1} \frac{z_1^k}{\Theta_1 \lceil j \rceil} = k\Theta_1^{\frac{1}{2}(\iota-3)} V_1^{k-1} < k\Theta_1^{\frac{1}{2}(\iota-3)} V_{1cr}^{k-1} .$$

Now $V_{1cr} < 1$, and Θ_1 is bounded, so the series $\sum_k \Theta_1^{\frac{1}{2}(\iota-3)} kV_{1cr}^{k-1}$ is convergent, which implies uniform convergence for the series for $\partial I_\iota / \partial z_1$.

Consider now the series for $\partial I^{(i,j)} / \partial z_k$ ($k = 2, 3, 4$) and for $\partial I^{(i,j)} / \partial \Theta_0$.

Uniform convergence in the series for $\partial I^{(i,j)} / \partial z_2$ is demonstrated by the observation that, apart from a bounded multiplier, the general term is

$$A_k(z_2) = \frac{\binom{j}{k}\left(\frac{z_3}{\Theta_0} - 1\right)^k}{(iz_z + kz_4 + 1)^2},$$

and so

$$|A_k(z_2)| < \left|\binom{j}{k}\right| \left(1-\frac{z_3}{\Theta_0}\right)^k .$$

Now $z_3/\Theta_0 < 1$; the method used to demonstrate the convergence of the series for $u^i\Theta^j$ shows that the series for $\partial I^{(i,j)}/\partial z_2$ for $z_2 > 0$ converges uniformly. Uniform convergence of the series for $\partial^{(i,j)}/\partial z_4$ is demonstrated in the same way.

Since $\Theta_0 \geq 1$ and $V_1 < 1$,

$$0 < 1 - \frac{z_3}{\Theta_0} < 1 - V_{1\,min} < 1,$$

so it is easy to show that the series for $\partial I^{(i,j)}/\partial z_3$ and $\partial I^{(i,j)}/\partial \Theta_0$ converge uniformly.

There remain the series $\partial I_\iota/\partial z_k$ ($\iota = 1, 2, 3$; $k = 2, 3, 4$) and $\partial I_\iota/\partial \Theta_0$ ($\iota = 1, 2, 3$). The proofs in all cases are analogous. For instance, the uniform convergence of $\partial I_\iota/\partial z_2$ is demonstrated as follows; this series consists of terms of the form $\partial I^{(i,j)}/\partial z_2$.

Now

$$I^{(i,j)} = z_1^i \int_0^1 y^{z_2 i} \Theta^j dy,$$

so from the theorem on the mean

$$I^{(i,j)} = \frac{z_1^i}{\overline{\Theta}|j|} \int_0^1 y^{z_2 i} dy = \frac{z_1^i}{\overline{\Theta}|j|} \frac{1}{z_2 i + 1} ,$$

in which $\overline{\Theta}$ is not dependent on z_2, so

$$\left| \frac{\partial I^{(i,j)}}{\partial z_2} \right| = \frac{i z_1^i}{\overline{\Theta}|j|(z_2 i + 1)^2} < \frac{i z_1^i}{\Theta|j|} = i \Theta_1^{\frac{1}{2}(i-2)} V_1 ,$$

since $z_2 > 0$ and $\overline{\Theta} \geq \Theta_1$. Also, $V_1 \leq V_{1cr} < 1$ and Θ_1 is bounded, so the series for the $\partial I_i / \partial z_2$ converge uniformly.

APPENDIX II

ESTIMATE OF THE ERRORS ARISING FROM APPROXIMATE CALCULATION OF THE INTEGRALS IN CHAPTERS II AND III.

Integrals I_1, I_2, and I_3 (chapter II) and I_u, I_{uu}, I_Θ, and $I_{u\Theta}$ (chapter II) were calculated on the assumption that power-law profiles can be used right out to the wall; no account was taken of the difference between power-law and linear profiles. We show here that the resulting errors are small. For simplicity we consider only the case of adiabatic flow with the velocity profile given in the form (see chapter III)

$$u = u_1 H^n .$$

With allowance for the sublayer, the profile may be put in the form

$$u = \begin{cases} u_1 H_\delta^{n-1} H, & 0 \le H \le H_\delta, \\ u_1 H^n, & H_\delta \le H \le 1. \end{cases}$$

We denote by I_u' and I_{uu}' the integrals analogous to I_u and I_{uu} but calculated with allowance for the sublayer. We have

$$I_u' = u_1 \int_0^{H_\delta} H_\delta^{n-1} HdH + u_1 \int_{H_\delta}^1 H^n a = u_1 \left(\frac{H_\delta^{n+1}}{2} + \frac{1-H_\delta^{n+1}}{n+1} \right),$$

whence

$$I'_u = I_u \left[1 - \frac{1}{2}(1 - n)\, H_\delta^{n+1}\right].$$

Similarly we have

$$I'_{uu} = I_{uu} \left[1 - \frac{2}{3}(1 - n) H_\delta^{2n+1}\right].$$

Then I_u and I_{uu} differ from I'_u and I'_{uu} by quantities whose order of smallness is not less than that of H_δ.

The difference in the derivatives with respect to u_1 is of the same order, because

$$\frac{\partial I'_u}{\partial u_1} = \frac{\partial I_u}{\partial u_1}\left[1 - \frac{1}{2}(1-n)H_\delta^{n+1}\right],$$

$$\frac{\partial I'_{uu}}{\partial u_1} = \frac{\partial I_u}{\partial u_1}\left[1 - \frac{2}{3}(1-n)H_\delta^{2n+1}\right].$$

In section 6 of chapter III we found that

$$H_\delta = \left(\frac{157}{G_1 \tilde{R}e}\ \frac{1}{n + 1}\right)^{\frac{1}{n+1}}$$

so

$$\frac{\partial}{\partial n} H_\delta^{n+1} = - \frac{H_\delta^{n+1}}{n+1} ,$$

$$\frac{\partial}{\partial n} H_\delta^{2n+1} = H_\delta^{2n+1}\left[\frac{\ln H_\delta}{n+1} - \frac{2n+1}{(n+1)^2}\right]$$

This implies that $\partial I'_u/\partial n$ and $\partial I'_{uu}/\partial n$ differ from $\partial I_u/\partial n$ and $\partial I_{uu}/\partial n$ by quantities of the order of H_δ and $H_\delta \ln H_\delta$.

APPENDIX III

Tables of Gas-Dynamic Functions for Air (k = 1.4)

M_*	τ	Π	g	Y	X	M
0.00	1.0000	1.0000	0.0000	0.0000	∞	0.0000
0.01	1.0000	0.9999	0.0158	0.0158	8562.105	0.0091
0.02	0.9999	0.9998	0.0315	0.0316	2135.793	0.0183
0.03	0.9999	0.9995	0.0473	0.0473	946.116	0.0274
0.04	0.9997	0.9990	0.0631	0.0631	530.107	0.0365
0.05	0.9996	0.9986	0.0788	0.0789	337.665	0.0457
0.06	0.9994	0.9979	0.0945	0.0947	233.250	0.0548
0.07	0.9992	0.9971	0.1102	0.1105	170.354	0.0639
0.08	0.9989	0.9963	0.1259	0.1263	129.534	0.0731
0.09	0.9987	0.9953	0.1415	0.1422	101.711	0.0822
0.10	0.9983	0.9942	0.1571	0.1580	81.753	0.0914
0.11	0.9980	0.9929	0.1726	0.1739	67.039	0.1005
0.12	0.9976	0.9916	0.1882	0.1897	55.876	0.1097
0.13	0.9972	0.9901	0.2036	0.2056	47.220	0.1190
0.14	0.9967	0.9886	0.2190	0.2216	40.354	0.1280
0.15	0.9963	0.9870	0.2344	0.2375	34.833	0.1372
0.16	0.9957	0.9851	0.2497	0.2535	30.333	0.1460
0.17	0.9952	0.9832	0.2649	0.2695	26.614	0.1560
0.18	0.9946	0.9812	0.2801	0.2855	23.508	0.1650
0.19	0.9940	0.9791	0.2952	0.3015	20.892	0.1740

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[Table continued]

M_*	τ	Π	g	Y	X	M
0.20	0.9933	0.9768	0.3102	0.3176	18.665	0.1830
0.21	0.9927	0.9745	0.3252	0.3337	16.761	0.1920
0.22	0.9919	0.9720	0.3401	0.3499	15.110	0.2020
0.23	0.9912	0.9695	0.3549	0.3660	13.678	0.2109
0.24	0.9904	0.9668	0.3696	0.3823	12.431	0.2202
0.25	0.9896	0.9640	0.3842	0.3985	11.336	0.2290
0.26	0.9887	0.9611	0.3987	0.4148	10.366	0.2387
0.27	0.9879	0.9581	0.4131	0.4311	9.514	0.2480
0.28	0.9869	0.9550	0.4274	0.4475	8.753	0.2573
0.29	0.9860	0.9518	0.4416	0.4640	8.068	0.2670
0.30	0.9850	0.9485	0.4557	0.4804	7.458	0.2760
0.31	0.9840	0.9451	0.4697	0.4970	6.913	0.2850
0.32	0.9829	0.9415	0.4835	0.5135	6.416	0.2947
0.33	0.9819	0.9379	0.4972	0.5302	5.969	0.3040
0.34	0.9807	0.9342	0.5109	0.5469	5.565	0.3134
0.35	0.9796	0.9303	0.5243	0.5636	5.196	0.3228
0.36	0.9784	0.9265	0.5377	0.5804	4.861	0.3322
0.37	0.9772	0.9224	0.5509	0.5973	4.556	0.3417
0.38	0.9759	0.9183	0.5640	0.6142	4.276	0.3511
0.39	0.9747	0.9141	0.5769	0.6312	4.020	0.3606
0.40	0.9733	0.9097	0.5897	0.6482	3.785	0.3701
0.41	0.9720	0.9053	0.6024	0.6654	3.602	0.3796
0.42	0.9706	0.9008	0.6149	0.6826	3.417	0.3892
0.43	0.9692	0.8962	0.6272	0.6998	3.233	0.3987
0.44	0.9677	0.8915	0.6394	0.7172	3.048	0.4083

[Table continued next page]

[Table continued]

M_*	τ	Π	g	Y	X	M
0.45	0.9663	0.8868	0.6515	0.7346	2.864	0.4179
0.46	0.9647	0.8819	0.6633	0.7521	2.739	0.4275
0.47	0.9632	0.8770	0.6750	0.7697	2.614	0.4372
0.48	0.9616	0.8719	0.6865	0.7874	2.490	0.4468
0.49	0.9600	0.8668	0.6979	0.8052	2.365	0.4565
0.50	0.9583	0.8616	0.7091	0.8230	2.240	0.4663
0.51	0.9567	0.8563	0.7201	0.8409	2.154	0.4760
0.52	0.9549	0.8509	0.7309	0.8590	2.068	0.4858
0.53	0.9532	0.8455	0.7416	0.8771	1.981	0.4956
0.54	0.9514	0.8400	0.7520	0.8953	1.895	0.5054
0.55	0.9496	0.8344	0.7623	0.9136	1.809	0.5152
0.56	0.9477	0.8287	0.7724	0.9321	1.748	0.5251
0.57	0.9459	0.8230	0.7823	0.9506	1.687	0.5350
0.58	0.9439	0.8172	0.7920	0.9692	1.627	0.5450
0.59	0.9420	0.8112	0.8015	0.9880	1.566	0.5549
0.60	0.9400	0.8053	0.8109	1.0069	1.505	0.5649
0.61	0.9380	0.7992	0.8198	1.0258	1.462	0.5750
0.62	0.9359	0.7932	0.8288	1.0449	1.419	0.5850
0.63	0.9339	0.7870	0.8375	1.0641	1.376	0.5951
0.64	0.9317	0.7808	0.8459	1.0842	1.333	0.6053
0.65	0.9296	0.7745	0.8543	1.1030	1.290	0.6154
0.66	0.9274	0.7681	0.8623	1.1226	1.260	0.6256
0.67	0.9252	0.7617	0.8701	1.1423	1.229	0.6359
0.68	0.9229	0.7553	0.8778	1.1622	1.199	0.6461
0.69	0.9207	0.7488	0.8852	1.1822	1.168	0.6565

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M_*	τ	Π	g	Y	X	M
0.70	0.9183	0.7422	0.8924	1.2024	1.138	0.6668
0.71	0.9160	0.7356	0.8993	1.2227	1.117	0.6772
0.72	0.9136	0.7289	0.9061	1.2431	1.095	0.6876
0.73	0.9112	0.7221	0.9126	1.2637	1.074	0.6981
0.74	0.9087	0.7154	0.9189	1.2845	1.052	0.7086
0.75	0.9063	0.7086	0.9250	1.3054	1.031	0.7192
0.76	0.9037	0.7017	0.9308	1.3265	1.016	0.7298
0.77	0.9012	0.6948	0.9364	1.3478	1.001	0.7404
0.78	0.8986	0.6878	0.9418	1.3692	0.987	0.7511
0.79	0.8960	0.6809	0.9469	1.3908	0.972	0.7619
0.80	0.8933	0.6738	0.9518	1.4126	0.957	0.7727
0.81	0.8907	0.6668	0.9565	1.4346	0.947	0.7835
0.82	0.8879	0.6597	0.9610	1.4567	0.937	0.7944
0.83	0.8852	0.6526	0.9652	1.4790	0.928	0.8053
0.84	0.8824	0.6454	0.9691	1.5016	0.918	0.8163
0.85	0.8796	0.6382	0.9729	1.5243	0.908	0.8274
0.86	0.8767	0.6310	0.9764	1.5473	0.902	0.8384
0.87	0.8739	0.6238	0.9796	1.5704	0.896	0.8496
0.88	0.8709	0.6165	0.9826	1.5938	0.889	0.8608
0.89	0.8680	0.6092	0.9854	1.6174	0.883	0.8721
0.90	0.8650	0.6019	0.9879	1.6412	0.877	0.8833
0.91	0.8620	0.5946	0.9902	1.6652	0.874	0.8947
0.92	0.8589	0.5873	0.9923	1.6895	0.871	0.9062
0.93	0.8559	0.5800	0.9941	1.7140	0.868	0.9177
0.94	0.8527	0.5726	0.9957	1.7388	0.865	0.9292

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M_*	τ	Π	g	Y	X	M
0.95	0.8496	0.5653	0.9970	1.7638	0.862	0.9409
0.96	0.8464	0.5579	0.9981	1.7891	0.861	0.9526
0.97	0.8432	0.5505	0.9989	1.8146	0.860	0.9644
0.98	0.8399	0.5431	0.9953	1.8404	0.859	0.9761
0.99	0.8367	0.5357	0.9999	1.8665	0.858	0.9880
1.00	0.8333	0.5283	1.0000	1.8929	0.857	1.0000
1.01	0.8300	0.5209	0.9999	1.9195	0.858	1.0120
1.02	0.8266	0.5135	0.9995	1.9464	0.859	1.0241
1.03	0.8232	0.5061	0.9989	1.9737	0.859	1.0363
1.04	0.8197	0.4987	0.9980	2.0013	0.860	1.0486
1.05	0.8163	0.4913	0.9969	2.0291	0.861	1.0609
1.06	0.8127	0.4840	0.9957	2.0573	0.863	1.0733
1.07	0.8092	0.4766	0.9941	2.0858	0.865	1.0858
1.08	0.8056	0.4693	0.9924	2.1147	0.868	1.0985
1.09	0.8020	0.4619	0.9903	2.1439	0.870	1.1111
1.10	0.7983	0.4546	0.9880	2.1734	0.872	1.1239
1.11	0.7947	0.4473	0.9856	2.2034	0.875	1.1367
1.12	0.7909	0.4400	0.9829	2.2337	0.878	1.1496
1.13	0.7872	0.4328	0.9800	2.2643	0.882	1.1627
1.14	0.7834	0.4255	0.9768	2.2954	0.885	1.1758
1.15	0.7796	0.4184	0.9735	2.3269	0.888	1.1890
1.16	0.7757	0.4111	0.9698	2.3588	0.892	1.2023
1.17	0.7719	0.4040	0.9659	2.3911	0.896	1.2157
1.18	0.7679	0.3969	0.9620	2.4238	0.900	1.2292
1.19	0.7640	0.3898	0.9577	2.4570	0.904	1.2428

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M_*	τ	Π	g	Y	X	M
1.20	0.7600	0.3827	0.9531	2.4906	0.908	1.2566
1.21	0.7560	0.3757	0.9484	2.5247	0.913	1.2708
1.22	0.7519	0.3687	0.9435	2.5593	0.917	1.2843
1.23	0.7478	0.3617	0.9384	2.5944	0.922	1.2974
1.24	0.7437	0.3548	0.9331	2.6300	0.926	1.3126
1.25	0.7396	0.3479	0.9275	2.6660	0.931	1.3268
1.26	0.7354	0.3411	0.9217	2.7026	0.936	1.3413
1.27	0.7312	0.3343	0.9159	2.7398	0.941	1.3558
1.28	0.7269	0.3275	0.9096	2.7775	0.947	1.3705
1.29	0.7227	0.3208	0.9033	2.8158	0.952	1.3853
1.30	0.7183	0.3142	0.8969	2.8547	0.957	1.4002
1.31	0.7140	0.3075	0.8901	2.8941	0.963	1.4153
1.32	0.7096	0.3010	0.8831	2.2343	0.968	1.4305
1.33	0.7052	0.2945	0.8761	2.9750	0.974	1.4458
1.34	0.7007	0.2880	0.8688	3.0164	0.979	1.4613
1.35	0.6962	0.2816	0.8614	3.0586	0.985	1.4769
1.36	0.6917	0.2753	0.8538	3.1013	0.991	1.4927
1.37	0.6872	0.2690	0.8459	3.1448	0.997	1.5087
1.38	0.6826	0.2628	0.8380	3.1889	1.002	1.5248
1.39	0.6780	0.2566	0.8299	3.2340	1.008	1.5410
1.40	0.6733	0.2505	0.8216	3.2798	1.014	1.5575
1.41	0.6687	0.2445	0.8131	3.3263	1.020	1.5741
1.42	0.6639	0.2385	0.8046	3.3737	1.026	1.5909
1.43	0.6592	0.2326	0.7958	3.4219	1.033	1.6078
1.44	0.6544	0.2267	0.7869	3.4710	1.039	1.6250

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M_*	τ	Π	g	Y	X	M
1.45	0.6496	0.2209	0.7778	3.5211	1.045	1.6423
1.46	0.6447	0.2152	0.7687	3.5720	1.051	1.6598
1.47	0.6398	0.2095	0.7593	3.6240	1.057	1.6776
1.48	0.6349	0.2040	0.7499	3.6768	1.064	1.6955
1.49	0.6300	0.1985	0.7404	3.7308	1.070	1.7137
1.50	0.6250	0.1930	0.7307	3.7858	1.076	1.7321
1.51	0.6200	0.1876	0.7209	3.8418	1.082	1.7506
1.52	0.6149	0.1824	0.7110	3.8990	1.089	1.7694
1.53	0.6099	0.1771	0.7009	3.9574	1.095	1.7885
1.54	0.6047	0.1720	0.6909	4.0172	1.102	1.8078
1.55	0.5996	0.1669	0.6807	4.0778	1.108	1.8273
1.56	0.5944	0.1619	0.6703	4.1398	1.114	1.8471
1.57	0.5892	0.1570	0.6599	4.2034	1.121	1.8672
1.58	0.5839	0.1522	0.6494	4.2680	1.127	1.8875
1.59	0.5786	0.1474	0.6389	4.3345	1.134	1.9081
1.60	0.5733	0.1427	0.6282	4.4020	1.140	1.9290
1.61	0.5680	0.1381	0.6175	4.4713	1.147	1.9501
1.62	0.5626	0.1336	0.6067	4.5422	1.153	1.9716
1.63	0.5572	0.1291	0.5958	4.6144	1.160	1.9934
1.64	0.5517	0.1248	0.5850	4.6887	1.166	2.0155
1.65	0.5463	0.1205	0.5740	4.7647	1.173	2.0380
1.66	0.5407	0.1163	0.5630	4.8424	1.180	2.0607
1.67	0.5352	0.1121	0.5520	4.9221	1.186	2.0839
1.68	0.5296	0.1081	0.5409	5.0037	1.193	2.1073
1.69	0.5240	0.1041	0.5298	5.0877	1.199	2.1313

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M_*	T	Π	g	Y	X	M
1.70	0.5183	0.1003	0.5187	5.1735	1.206	2.1555
1.71	0.5126	0.0965	0.5075	5.3167	1.213	2.1802
1.72	0.5069	0.0928	0.4965	5.3520	1.219	2.2053
1.73	0.5012	0.0891	0.4852	5.4449	1.226	2.2308
1.74	0.4954	0.0856	0.4741	5.5403	1.232	2.2567
1.75	0.4896	0.0821	0.4630	5.6383	1.239	2.2831
1.76	0.4837	0.0787	0.452	5.7390	1.246	2.3100
1.77	0.4779	0.0754	0.4407	5.8427	1.252	2.3374
1.78	0.4719	0.0722	0.4296	5.9495	1.259	2.3653
1.79	0.4660	0.0691	0.4185	6.0593	1.265	2.3937
1.80	0.4600	0.0660	0.4075	6.1723	1.272	2.4227
1.81	0.4540	0.0630	0.3965	6.2893	1.279	2.4523
1.82	0.4479	0.0602	0.3855	6.4091	1.285	2.4824
1.83	0.4418	0.0573	0.3746	6.5335	1.292	2.5132
1.84	0.4357	0.0546	0.3638	6.6607	1.298	2.5449
1.85	0.4296	0.0520	0.3530	6.7934	1.305	2.5766
1.86	0.4234	0.0494	0.3423	6.9298	1.312	2.6094
1.87	0.4172	0.0469	0.3316	7.0707	1.318	2.6429
1.88	0.4109	0.0445	0.3211	7.2162	1.325	2.6772
1.89	0.4047	0.0422	0.3105	7.3673	1.331	2.7123
1.90	0.3983	0.0399	0.3002	7.5243	1.338	2.7481
1.91	0.3920	0.0377	0.2898	7.6858	1.344	2.7849
1.92	0.3856	0.0356	0.2797	7.8540	1.351	2.8225
1.93	0.3792	0.0336	0.2695	8.0289	1.357	2.8612
1.94	0.3727	0.0316	0.2596	8.2098	1.364	2.9007

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M_*	τ	Π	g	Y	X	M
1.95	0.3662	0.0297	0.2497	8.3985	1.370	2.9414
1.96	0.3597	0.0279	0.2400	8.5943	1.376	2.9831
1.97	0.3532	0.0262	0.2304	8.7984	1.383	3.0301
1.98	0.3466	0.0245	0.2209	9.0112	1.389	3.0701
1.99	0.3400	0.0229	0.2116	9.2329	1.396	3.1155
2.00	0.3333	0.0214	0.2024	9.464	1.402	3.1622
2.01	0.3267	0.0199	0.1934	9.706	1.408	3.2104
2.02	0.3199	0.0185	0.1845	9.961	1.415	3.2603
2.03	0.3132	0.0172	0.1758	10.224	1.421	3.3113
2.04	0.3064	0.0159	0.1672	10.502	1.428	3.3642
2.05	0.2996	0.0147	0.1588	10.794	1.434	3.4190
2.06	0.2927	0.0136	0.1507	11.102	1.440	3.4759
2.07	0.2859	0.0125	0.1427	11.422	1.447	3.5343
2.08	0.2789	0.0115	0.1348	11.762	1.453	3.5951
2.09	0.2720	0.0105	0.1272	12.121	1.460	3.6583
2.10	0.2650	0.0096	0.1198	12.500	1.466	3.7240
2.11	0.2580	0.0087	0.1125	12.901	1.472	3.7922
2.12	0.2509	0.0079	0.1055	13.326	1.479	3.8633
2.13	0.2439	0.0072	0.0986	13.778	1.485	3.9376
2.14	0.2367	0.0065	0.0921	14.259	1.492	4.0150
2.15	0.2296	0.0058	0.0857	14.772	1.498	4.0961
2.16	0.2224	0.0052	0.0795	15.319	1.504	4.1791
2.17	0.2152	0.0046	0.0735	15.906	1.510	4.2702
2.18	0.2079	0.0041	0.0678	16.537	1.517	4.3642
2.19	0.2006	0.0036	0.0623	17.218	1.523	4.4633

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M_*	τ	Π	g	Y	X	M
2.20	0.1933	0.0032	0.0570	17.949	1.529	4.5674
2.21	0.1860	0.0028	0.0520	18.742	1.535	4.6778
2.22	0.1786	0.0024	0.0472	19.607	1.541	4.7954
2.23	0.1712	0.0021	0.0427	20.548	1.547	4.9201
2.24	0.1637	0.0018	0.0384	22.983	1.553	5.0533
2.25	0.1563	0.00151	0.0343	22.712	1.559	5.1958
2.26	0.1487	0.00127	0.0304	23.968	1.565	5.3494
2.27	0.1412	0.00106	0.0268	25.361	1.571	5.5147
2.28	0.1336	0.00087	0.0234	26.893	1.578	5.6940
2.29	0.1260	0.00071	0.0204	28.669	1.584	5.8891
2.30	0.1183	0.00057	0.0175	30.658	1.590	6.1033
2.31	0.1106	0.00045	0.0148	32.937	1.596	6.3399
2.32	0.1029	0.00035	0.0124	35.551	1.602	6.6008
2.33	0.0952	0.00027	0.0103	38.606	1.608	6.8935
2.34	0.0874	0.00020	0.0083	42.233	1.614	7.2254
2.35	0.0796	0.00014	0.0063	46.593	1.620	7.6053
2.36	0.0717	$0.988\cdot10^{-4}$	0.0051	51.914	1.626	8.0450
2.37	0.0638	$0.657\cdot10^{-4}$	0.0038	58.569	1.632	8.5619
2.38	0.0559	$0.413\cdot10^{-4}$	0.0028	67.144	1.638	9.1882
2.39	0.0480	$0.242\cdot10^{-4}$	0.0019	78.613	1.644	9.9624
2.40	0.0400	$0.128\cdot10^{-4}$	0.0012	94.703	1.650	10.957
2.41	0.0320	$0.584\cdot10^{-5}$	0.0007	118.94	1.656	12.306
2.42	0.0239	$0.211\cdot10^{-5}$	0.0003	159.65	1.661	14.287
2.43	0.0158	$0.499\cdot10^{-6}$	0.0001	242.16	1.667	17.631
2.44	0.0077	$0.316\cdot10^{-7}$	$0.58\cdot10^{-4}$	499.16	1.672	25.367
2.449	0	0	0		1.678	

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